Tensor Operators and Mass Formula in the Minimal Extension of U_3 by Charge Conjugation*

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The transformation property of the mass (or mass square) M is specified in the extended group. The most general mass formula, $M = M_0 + a(\alpha)[I(I+1) - \frac{1}{4}Y^2 - K(\alpha)] + b_1(\alpha)NY + b_2(\alpha)Q_2Y$, is derived where $a(\alpha)$, $b_i(\alpha)$, $K(\alpha)$ are constants depending on a charge-even set α, specified in the text, and Q_2 is the second invariant operator of SU_3.

THE Gell-Mann–Okubo mass formula, $M = M_0 + a[I(I+1) - \frac{1}{4}Y^2] + bY$, is incomplete in the sense that it does not reflect, as it stands, the charge conjugation properties of the mass splitting. Thus, the last term should have opposite signs for baryons and antibaryons and should vanish for mesons. Therefore, the baryon number and charge conjugation parities must enter into the formula as well. In order to modify the form of the equation, we start from the invariance group $\mathcal{E} = \{U_3, \text{coset } U_3C\}$, which is the minimal extension[1] of U_3 by charge conjugation C.[2]

I. THE STATES

The states of irreducible representations of U_3 are labeled by $|N, Q_1, Q_2; I, I_3, Y\rangle$, where Q_1 and Q_2 are the two invariant (Casimir) operators of SU_3. Note that Q_1 and Q_2 are chosen to be the components of the maximal weight of the representation:

$$Q_1 = \tfrac{1}{2}(p+q), \quad Q_2 = \frac{p-q}{2\sqrt{3}}.$$

Since $CpC = q$ and $CqC = p$, we find $CQ_1C = Q_1$, $CQ_2C = -Q_2$. Also, $CNC = -N$. The quadratic Casimir operator is

$$\sum_{i=1}^{8} F_i^2 = \tfrac{1}{3}[p(p+3) + q(q+3) + pq].$$

The charge conjugation C can be adjoined to the representations of U_3 in two possible ways: either the states $C|\rangle$ are unrelated to the original states, then the extended group $\mathcal{E} = \{U_3, U_3C\}$ is represented in the Hilbert space $\begin{Bmatrix} |\rangle \\ C|\rangle \end{Bmatrix}$ of doubled states; or the states $C|\rangle$ are linearly related to the states $|\rangle$, then we can form the combinations $(2)^{-\frac{1}{2}}[|\rangle \pm C|\rangle]$ with definite C-parities η_c, which are denoted by

$$|\eta_c; N, Q_1, Q_2; I, I_3, Y\rangle, \quad \eta_c = \pm 1. \tag{1}$$

This second case occurs only if the invariant operators of U_3 are also invariant under C, that is, if $N = 0$, $Q_2 = 0$ in our choice of the Casimir operators (see note above) (i.e., self-adjoint representations of U_3).

II. THE INVARIANTS

The invariant operators out of which the most general invariant function is constructed is different in both cases. In the first case, where the states are doubled, the two representations of U_3 with $\pm N$, $\pm Q_2$ are the same irreducible representation of the extended group $\mathcal{E}$ characterized by N^2, Q_2^2. Thus, due to the additional requirement

$$Cf(N, Q_1, Q_2)C^{-1} = f(N, Q_1, Q_2), \tag{2}$$

all invariants are functions of only

$$N^2, Q_1, NQ_2, \text{ and } Q_2^2. \tag{3}$$

The operator NQ_2 fixes the relative sign between N and Q_2 to distinguish, for example, between

$$\begin{pmatrix} 10, N=1 \\ \overline{10}, N=-1 \end{pmatrix} \text{ and } \begin{pmatrix} \overline{10}, N=1 \\ 10, N=-1 \end{pmatrix};$$

only the first case is known to be realized for the $\frac{3}{2}^+$ baryons.

In the second case the invariants are functions of η_c and Q_1. Thus we can write in both cases the invariants as functions of the set

$$\alpha = \{\eta_c\delta_{N,0}, N^2, Q_1, NQ_2, Q_2^2\}. \tag{4}$$

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[1] The concept of minimal extension of a group by a discrete operation is discussed in detail in the articles by E. P. Wigner and L. Michel, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, F. Gürsey, Ed. (Gordon and Breach Science Publishers, Inc., New York, 1964); T. D. Lee and G. C. Wick, Phys. Rev. **148**, 1385 (1966).

[2] Mathematically the charge conjugation C defines an automorphism of U_3 and this automorphism characterizes the minimal extension. L. C. Biedenharn, J. Nuyts, and H. Ruegg, CERN preprint (1965); S. Okubo and N. Mukunda, Ann. Phys. (N.Y.) **36**, 311 (1966). Depending on whether the representation of the automorphism is inner or outer, one gets the two cases of doubling or no doubling of states discussed below.

III. THE TENSOR OPERATORS

The general tensor operator is first constructed for the group U_3 in the usual way from the infinitesimal generators F_i, $i = 0, 1, \cdots, 8$ in the form

$$T_i = t_i + t_{ij}F_j + t_{ijk}F_jF_k + \cdots, \quad (5)$$

where $t_{i_1 \ldots i_n}$ are all possible invariant symmetric tensors of the adjoint representation of U_3. This follows from

$$Ut_{ijk} \cdots F_jF_k \cdots U^{-1}$$

$$= t_{ijk} \cdots F_{j'}AdU_{j'j}F_{k'} \; AdU_{k'k} = t_{i'jk} \cdots F_jF_kAdU_{i'i}.$$

Hence

$$AdU_{j'j}AdU_{k'k}AdU_{i'i}t_{i'j'k'} = t_{ijk}.$$

Applying this to T_8, for example, one gets

$$T_8 = a[I(I+1) - \tfrac{1}{4}Y^2 - K] + bY, \quad (6)$$

where a, b, K are functions of the U_3-invariants N, Q_1, Q_2. The constant K has the effect of making the average of T_8 over a multiplet vanish and is given by

$$K = \tfrac{1}{3}\sum_{i=1}^{8} F_i^2 = \tfrac{1}{9}[p(p+3) + q(q+3) + pq]. \quad (7)$$

Then we impose the condition of charge invariance

$$CT_iC^{-1} = T_i. \quad (8)$$

IV. THE MASS FORMULA

We require that the mass splitting be invariant under C in addition to the T_8-property, i.e.,

$$CT_8C^{-1} = T_8. \quad (9)$$

(Note that $CF_8C^{-1} = -F_8$!) The condition (9) restricts the coefficient a in Eq. (6) to be a function only of the set α, Eq. (4); K remains the same because $CKC^{-1} = K$, while b has to be the most general odd function under C, hence

$$b = b_1(\alpha)N + b_2(\alpha)Q_2.$$

Therefore, the most general mass formula in $\mathcal{E} = \{U_3, U_3C\}$ under the stated assumptions is

$$M = M_0 + a(\alpha)[I(I+1) - \tfrac{1}{4}Y^2 - K] + b_1(\alpha)NY + b_2(\alpha)Q_2Y. \quad (10)$$

V. OTHER CONCLUSIONS

(i) Because the group SU_n for $n \geq 3$ has only one outer automorphism, we do not expect any further extension of the internal symmetry group except the one discussed.

(ii) In the case of $N = 0$, only one value of η_c is known at present. Because the coefficients in the mass formula (10) depend on η_c, the states with $\eta_c = -1$ could lie higher.

(iii) Note the presence of the term b_2Q_2Y in Eq. (10) which distinguishes, for example, the $N = 1$ octet and decouplet even if the coefficients $a_1b_i(\alpha)$ are the same for these two multiplets.