UNIVERSAL 1/*R* TERM IN QUARK POTENTIAL OF SPONTANEOUS STRINGS ☆

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Strings with extrinsic curvature stiffness are characterized by two parameters, the tension and the "normality" ν which specifies the ratio between ordinary Nambu–Goto tension and spontaneously generated tension (the case $\nu=\infty$ is the Nambu–Goto string itself). In the limit of infinite dimensions we show that for all ν the asymptotic $1/R_{\rm ext}$ piece in the quark potential has the universal strength $-[(d-2)/2]\pi/12R_{\rm ext}$.

A spontaneous string is defined by the action [1,2] [#1]

$$\begin{aligned}\mathscr{A}&=\mathscr{A}_{\rm NG}+\mathscr{A}_{\rm K}\\ &=M_{\rm NG}^2\int d^2\xi\,\sqrt{g}+\frac{1}{2\alpha}\int d^2\xi\,\sqrt{g}(D^2x^a)^2\\ &\equiv\frac{d-2}{2}\int d^2\xi\,\sqrt{g}\{\tilde{M}_{\rm NG}^2\\ &\quad+(1/2\tilde{\alpha})[(D^2x^a)^2+\lambda^{ij}(\partial_i x^a\partial_j x^a-g_{ij})]\}\,,\end{aligned}\qquad(1)$$

where $\mathscr{A}_{\rm NG}$ is the ordinary Nambu–Goto action and $\mathscr{A}_{\rm K}$ is proportional to the square of the extrinsic curvature [we use the notation of ref. [2] with $x^a(\xi)$ parametrizing the surface in d dimensions, g_{ij} being the metric $\partial_i x^a\partial_j x^a$ ($i=1,2$), and D_i the covariant derivatives]. In the limit $d\to\infty$, the action is dominated by the saddle point of the integrated quantity [4]

$$\mathscr{A}=\mathscr{A}_0+\frac{d-2}{2}\left(\mathrm{tr}\ln(D^4-D_i\lambda^{ij}D_j)-\int d^2\xi\,\sqrt{g}\lambda^{ij}g_{ij}\right),\qquad(2)$$

where $\mathscr{A}_0$ is the action of a background configuration $x_0(\xi)$. The static quark potential is obtained by choosing $x_0^i(\xi)=\xi^i$ ($i=1,2$) so that g_{ij} in the large brackets is simply replaced by $g_{ij}-\delta_{ij}$.

The spontaneous string seems to be a much better surface representation of the string between quarks than the ordinary string: Its action is asymptotically free at short distances and the behaviour at large distance is controlled by a string tension which contains a dimensionally transmuted coupling constant

$$\bar{\lambda}=\mu^2 4\pi\exp(-\gamma)\exp[-(4\pi/\tilde{\alpha}-1)]\,.$$

At finite distance, the saddle point of (2) has a constant $g_{ij}=\rho_i\delta_{ij}$, $\lambda^{ij}=\lambda_i g_{ij}$ with values obtained by extremizing

$$\mathscr{A}=[(d-2)/2]2\pi R_{\rm ext}\beta_{\rm ext}g\,,\qquad(3)$$

with

$$g=\sqrt{\rho_0\rho_1}\left(\tilde{M}_{\rm NG}^2+\int\frac{d^2q}{(2\pi)^2}[\ln(q^4+\lambda_0q_0^2+\lambda_1q_1^2)]-\tilde{\lambda}/\tilde{\alpha}+(1/2\tilde{\alpha})(\lambda_0/\rho_0+\lambda_1/\rho_1)\right)\qquad(4)$$

($q_i\equiv k_i/\sqrt{\rho_i}\equiv$ intrinsic momenta) which, after introducing the intrinsic dimensionally transmuted coupling constant $\bar{\lambda}$ and setting $\tilde{M}_{\rm NG}^2\equiv\bar{\lambda}_\nu\nu/4\pi\equiv\bar{\lambda}e^\nu\nu/4\pi$, becomes

$$g=\sqrt{\rho_0\rho_1}\{(\bar{\lambda}_\nu\nu/4\pi)\nu-(\tilde{\lambda}/4\pi)[\ln(\tilde{\lambda}/\bar{\lambda})+(1/2\tilde{\alpha})(\lambda_0/\rho_0+\lambda_1/\rho_1)]\}.\qquad(5)$$

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[#1] See also related previous work on biomembranes in ref. [3].

At the extremum, $\rho_0=\rho_1=\bar\rho=4\pi/\tilde\alpha$, $\lambda_0=\lambda_1=\bar\lambda_\nu$ and $g=\bar g=\bar\lambda_\nu(1+\nu)/4\pi$. The total string tension is $M^2_{\rm tot}=[(d-2)/2]\bar g$. In a thermal environment, the spontaneous string looses its tension at a deconfinement temperature of about $0.7M_{\rm tot}$ [5].

Several recent papers have attempted to calculate the quark potential $V(R)$ for arbitrary R. Olesen and Yang [6] have assumed the saddle point to be constant with $\lambda_{ij}=\lambda g_{ij}$ ("isotropic gap") at finite R and found that then $V(R)$ has, for $R\to\infty$, a $1/R$ ("Lüscher" [7]) term that depends on the normality ν, namely $-[(d-2)/2]c_\nu\pi/12R_{\rm ext}$ with $c_\nu=1-3\nu/2\times(1+\nu)^2$. The author has pointed out [8] that the gap parameters λ_0 and λ_1 are very different ("anisotropic gap") and that, for constant gaps, the parameter c_ν receives an additional $-12/(3+2\nu)$.

An important observation was made by Braaten et al. [9,10]. Since the end points of the string do not move, they satisfy $\partial_0 x^a=0$ $(a=1,...,d)$, and hence $g_{00}=1$, at $\xi^1=0$ and $R_{\rm ext}$ for all times. The gaps of refs. [5,6], however, have $g_{00}\equiv\rho_0\neq1$ and violate this condition. It is therefore necessary to allow for a ξ^1 dependent saddle point. Braaten et al. were able to find such a saddle point in the limiting case of infinite stiffness, $\alpha\to0$, with $\tilde M^2_{\rm NG}\alpha=$fixed. In that limit they find that c_ν has the value unity, as for the pure Nambu–Goto string. From the fact that $c=1$ in two opposite limits the authors conclude that "it appears very unlikely that there are corrections of any kind to the Lüscher term". The reasoning, however, is not entirely correct. It can easily be checked that with $\tilde M^2_{\rm NG}=\bar\lambda_\nu\nu/4\pi$ and $\bar\lambda$ expressed in terms of μ^2 and $\tilde\alpha$ as given above, their limit corresponds to $\nu\approx4\pi/\tilde\alpha\to\infty$, in which case the Lüscher term becomes blind for the space dependence of the gaps. Indeed, our formula for c_ν gives also $c=1$ and a non-trivial ν-dependence of c_ν is not at all ruled out. In addition, refs. [9,10] contain a mistake in the renormalization [#2] which leads to an unphysical behaviour of the surface ratio intrinsic/extrinsic (see the end of section 3 in ref. [10]).

Since a universal c would be theoretically much more appealing than a ν dependent one, it is important to use space-dependent gaps and calculate c_ν for all ν. This is what we want to do in this note and the result is, indeed, $c_\nu\equiv1$. Physically, the reason turns out to be quite simple: For large R, the gap which at the ends has $\rho_0=1$, increases rapidly over a length scale $\xi_{\rm p}\equiv1/\sqrt{\bar\lambda_\nu}$ ($\equiv$persistence length) to a practically constant value whose magnitude is logarithmically divergent in the short-distance cutoff Λ^{-1}, like $[1/(1+\nu)][\ln(\Lambda^2/\bar\lambda)-1]$, and does not have any first-order variation in $1/R$. The large-R potential is therefore that of an ordinary string of a slightly reduced length (by $\approx2\xi_{\rm p}$). this is why $c_\nu\equiv1$. In order to show this we proceed as follows. First we expand g around the $R=\infty$ configuration allowing for smooth quadratic deviations in $\rho_{i,j}\equiv\bar\rho(1+r_{i,j})$, $\lambda_{i,j}\equiv\bar\lambda_\nu(1+l_{i,j})$. (In contrast to ref. [8] we shall not renormalize $\bar\rho$, $\bar\lambda_\nu$ down to finite values by a factor $\ln(\Lambda^2/\mu^2)$.) This gives [#3] $g=\bar g+\Delta g$ with

$$\Delta g=\bar g\int_0^{2\pi}\frac{d\vartheta}{2\pi}\{\tfrac12 r_-^2-r_+l_+-r_-l_-$$

$$-[1/2(1+\nu)]l_+^2-[1/2(1+\nu)]l_-^2\}+\Delta g^{\rm R}, \qquad (6)$$

where $\vartheta\equiv2\pi\xi^1/R_{\rm ext}$, $r_\pm$ and $l_\pm$ are the combinations $\frac12(0\pm1)$ of r_i, l_i, and

$$\Delta g^{\rm R}=(\bar\lambda_\nu/\pi)S_1$$

with

$$S_1\equiv\frac{1}{\bar\lambda_{\rm R}}\left(\sum_{n=1}^{\infty}-\int_0^\infty dn\right)\sqrt{n^2+\bar\lambda_{\rm R}}$$

(with $\bar\lambda_{\rm R}\equiv\bar\lambda_\nu\rho_1R^2_{\rm ext}/\pi^2$) containing the finite R corrections. The leading terms are

$$\Delta g_0^{\rm R}=\bar g(-2/\sqrt{\bar\lambda_{\rm R}}-1/3\bar\lambda_{\rm R})/(1+\nu)\ ,$$

of which the first is an additive constant in $V(R)$, and the second the correct Lüscher term. So, what we want to prove is the absence of more $1/\sqrt{\bar\lambda_{\rm R}}$ contributions. Most dangerous are the linear terms in $r_\pm$, $l_\pm$ which read

$$\Delta g_1^{\rm R}=\frac{\bar g}{1+\nu}\int_0^{2\pi}\frac{d\vartheta}{2\pi}(R_+r_++R_-r_-+L_+l_++L_-l_-)\ .$$

[#2] The proper covariant treatment of $(D^2)^2$ as in ref. [5] would lead to a gap equation (13) of ref. [9] and (3.11) of ref. [10] with no $\ln(1+\sigma)$ term. The error is of the same kind as that in ref. [11] which there it had so dramatic consequences that they were noted soon. (See ref. [12].)

[#3] Braaten et al. choose their limit so that they can ignore the $l_\pm^2$ terms which, as we see from the properly renormalized equation (6), amounts to taking the limit $\nu\to\infty$.

Under the assumption of a constant saddle point we found the coefficients [8] $L_+ = 4S_1 - 2S_2$, $L_- = 2S_2$, $R_+ = 2S_2$, $R_- = 2(S_1 - S_2) + R_+$ where S_2 denotes the sum

$$S_2 \equiv \left(\sum_{n=1}^{\infty} - \int_0^{\infty} \mathrm{d}n \right) 1/\sqrt{n^2 + \bar{\lambda}_R} \, .$$

An expansion in powers of $1/\sqrt{\bar{\lambda}_R}$ proceeds via the Euler–McLaurin formula $\sum_1^\infty f(n) = \sum_0^\infty \zeta(-k) \times f^{(k)}(k)/k!$ ($\zeta(z) \equiv$ Riemann's zeta function). In addition there are $\exp(-\sqrt{\bar{\lambda}_R})$ terms. Altogether:

$$S_1 = -1/2\sqrt{\bar{\lambda}_R} - 1/12\bar{\lambda}_R - 2\sum_{\tilde{n}=1}^{\infty} K_1(z_{\tilde{n}})/z_{\tilde{n}} \, ,$$

$$S_2 = -1/2\sqrt{\bar{\lambda}_R} + 2\sum^{\infty} K_0(z_{\tilde{n}})$$

($z_{\tilde{n}} \equiv 2\pi\sqrt{\bar{\lambda}_R}\tilde{n}$, $K_{0,1}(z)$ are Bessel functions.) Minimizing Δg in $l_\pm$, $r_\pm$ gives

$$l_+ = R_+/(1+\nu) \, ,$$

$$l_- = 2/(3+2\nu)(R_- + L_-) \, ,$$

$$r_+ = L_+/(1+\nu) - R_+/(1+\nu)^2 \, ,$$

$$r_- = [1/(3+2\nu)][2L_- - R_-/(1+\nu)] \, ,$$

so that we obtain the $1/R$ expansion

$$\Delta g = \bar{g}\frac{1}{1+\nu}\left[-\frac{2}{\sqrt{\bar{\lambda}_R}} - \left(\frac{1}{3\bar{\lambda}_R} + \int_0^{2\pi} \frac{\mathrm{d}\vartheta}{2\pi} \frac{1}{2(1+\nu)} \right.\right.$$
$$\times [R_+^2/(1+\nu) + R_-^2 - 2R_+L_+$$
$$\left.\left. -2(R_- + L_-)^2(1+\nu)/(3+2\nu)] \right) \right] .$$

Since $R_\pm$, $L_\pm$ all go like $-1/\sqrt{\bar{\lambda}_R}$, the large bracket yields c_ν as calculated in ref. [8] (and stated above). Let us now admit ξ^1 dependent gaps. Then (5) receives a derivative term $(2/\bar{\lambda}_R)(\partial_\vartheta r_-)^2$ from $\sqrt{g}(\mathrm{D}^2\xi^i)^2$ plus similar and higher derivative terms for all $r_\pm$, $l_\pm$ from higher loop diagrams – involving $\langle x^a x^a \rangle$. The sums $S_{1,2}$ are modified by a factor $(1 - \cos(\vartheta n))$, where the sums with the cosines do not have the $\int \mathrm{d}n$'s subtracted. This changes drastically the large-R limits. The leading behaviour comes now entirely from the $\int \mathrm{d}n$'s and is

$$L_- \to -2K_0(z)/(1+\nu) \, ,$$

$$L_+ \to [2K_0(z) + 4K_1(z)/z - 4/z^2]/\nu \, ,$$

$$R_+ \to -2K_0(z)/(1+\nu) \, ,$$

$$R_- \to [-1/3\bar{\lambda}_R - 2K_0(z)]/(1+\nu)$$

$$(z \equiv \vartheta\sqrt{\bar{\lambda}_R}\tilde{n}) \, .$$

Hence, at any fixed-distance ratio ϑ from the ends, the deviations from the asymptotic gaps, $r_\pm$, $l_\pm$ are exponentially small. Only at the ends there is a logarithmic divergence. In a proper lattice formulation of the short-distance cutoff Λ^{-1} would have to bring the value of ρ_0 precisely down to $\rho_0 = 1$. In the present approximation, the solution overshoots.

The $1/\sqrt{\bar{\lambda}_R}$ terms are cancelled since the factor $(1 - \cos(\vartheta n))$ in $f(n)$ deletes the $\zeta(0)f(0)$ term in the Euler–McLaurin sries. The first correction is of order $1/\bar{\lambda}_R$. Doing the ϑ integral in (8) can give now only $1/\sqrt{\bar{\lambda}_R}$ terms with no $1/\bar{\lambda}_R$ correction. We therefore find, up to this point, $c_\nu \equiv 1$ for all ν.

In order to complete the proof we have to make sure that the coefficient of all the omitted higher powers in $r_\pm$, $l_\pm$ and their gradients do not have any $1/\sqrt{\bar{\lambda}_R}$ corrections. This is somewhat tedious to show. Basically it follows from the fact that the integrals over all higher powers of $K_0(z)$ contribute to order $1/\sqrt{\bar{\lambda}_R}$ and so do the integrals involving derivatives of $K_0(z)$, which, to leading order in R_{ext}, all appear in the form $\xi_p\partial$. The next $1/R_{\mathrm{ext}}$ corrections carry, in all cases, two more powers in $1/R_{\mathrm{ext}}$ so that they cannot change the $1/R_{\mathrm{ext}}$ term in the quark potential.

This proves the universality of the number $c_\nu \equiv 1$ for spontaneous strings of any normality ν.

References

[1] A.M. Polyakov, Nucl. Phys. B 268 (1986) 406.
[2] H. Kleinert, Phys. Lett. B 174 (1986) 335.
[3] W. Helfrich, J. Phys. (Paris) 46 (1985) 1263;
L. Peliti and S. Leibler, Phys. Rev. Lett. 56 (1985) 1690;
D. Förster, Phys. Lett. A 114 (1986) 115;
H. Kleinert, Phys. Lett. a 114 (1986) 263; A 116 (1986) 57;
S. Ami and H. Kleinert, Phys. Lett. A 120 (1987) 207;
W. Janke and H. Kleinert, Phys. Rev. Lett. 58 (1987) 144.
[4] F. David, Europhys. Lett. 2 (1987) 577;
F. David and E. Guitter, Europhys. Lett., to be published;
A. Alonso and D. Espriu, Nucl. Phys., to be published.
[5] H. Kleinert, Phys. Lett. B 189 (1987) 187.
[6] P. Olesen and S.-K. Yang, Nucl. Phys. B 283 (1987) 73.

[7] M. Lüscher, Nucl. Phys. B 180 (1981) 317;
J.D. Stack and M. Stone, Phys. Lett. B 100 (1981) 476.
[8] H. Kleinert, Phys. Rev. Lett. 58 (1987) 1915.
[9] E. Braaten, R.D. Pisarski and S.-M. Tse, Phys. Rev. Lett. 58 (1987) 93.
[10] E. Braaten and S.-M. Tse, Argonne preprint.
[11] R. Pisarski, Phys. Rev. Lett. 58 (1987) 1300.
[12] R. Pisarski, Erratum to Phys. Rev. Lett. 58 (1987) 1300; Fermilab preprint (revised).