Double-Gauge Invariant Local Quantum Field Theory of Charges and Dirac Magnetic Monopoles *

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Abstract

Exploiting the recently found extra *monopole gauge symmetry* which ensures the physical irrelevance of the Dirac strings in electromagnetism with Dirac magnetic monopoles, we formulate a local quantum field theory of charges and monopoles.

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1) In a recent note [1] we have pointed out the existence of an extra gauge symmetry in the action describing the electromagnetic forces between a particle of charge e on a worldline L' with a 4-current

$$j^\mu = e\delta_\mu(x; L'), \tag{1}$$

and a Dirac magnetic monopole [2] of charge g on a worldline L with a current

$$\tilde{j}^\mu = g\delta_\mu(x; L). \tag{2}$$

The action reads

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_{el} \equiv \int d^4x \left[\frac{1}{16\pi}(F_{\mu\nu} - F^P_{\mu\nu})^2 + iA_\mu j_\mu\right], \tag{3}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength while $F^P_{\mu\nu}$ is what we have called the *monopole gauge field*, describing the monopole via the dual

$$\tilde{\delta}_{\mu\nu}(x; S) \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\delta_{\lambda\kappa}(x; S), \tag{4}$$

of the δ-function on the worldsheet S of the Dirac string

$$\delta_{\mu\nu}(x; S) \equiv \int d\sigma d\tau \left[\frac{d\bar{x}_\mu(\sigma)}{d\sigma}\frac{d\bar{x}_\nu(\tau)}{d\tau} - (\mu \leftrightarrow \nu)\right]\delta^{(4)}(x - \bar{x}(\sigma,\tau)), \tag{5}$$

in the following way:

$$F^P_{\mu\nu} \equiv 4\pi g\tilde{\delta}_{\mu\nu}(x; S). \tag{6}$$

The physically observable field strength is $F^{obs}_{\mu\nu} = F_{\mu\nu} - F^P_{\mu\nu}$ and the finiteness of the action $\mathcal{A}_1$ containing the monopole gauge field enforces the presence of a δ-function in $F_{\mu\nu}$ on the world surface precisely equal to $F^P_{\mu\nu}$ so that $F^{obs}_{\mu\nu}$. This is why the action $\mathcal{A}_1$ is regular and does not contain a square of a δ-function as the expression (3) might initially suggest.

The worldline of the monopole is, of course, the boundary line of the string's worldsheet S, as expressed by Stokes' theorem:

$$\frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\partial_\nu\tilde{\delta}_{\lambda\kappa}(x; S) = \delta_\mu(x; L). \tag{7}$$

This implies that the monopole gauge field satisfies the equation

$$\frac{1}{2}\epsilon_{\mu\nu\lambda\kappa}\partial_\nu F^P_{\lambda\kappa} = 4\pi \tilde{j}_\mu. \tag{8}$$

2) As noted in [1], the action (3) is invariant under the *monopole gauge transformations*

$$F^P_{\mu\nu} \to F^P_{\mu\nu} + \partial_\mu \Lambda^P_\nu - \partial_\nu \Lambda^P_\mu, \qquad A_\mu \to A_\mu + \Lambda^P_\mu, \tag{9}$$

with integrable vector functions $\Lambda^P_\mu(x)$, which have the general form

$$\Lambda^P_\mu = 4\pi g \sum_V \delta_\mu(x;V), \tag{10}$$

with the sum running over arbitrary choices of 3-volumes V and $\delta_\mu(V)$ being the δ-function on these volumes:

$$\delta_\mu(x;V) \equiv \epsilon_{\mu\nu\kappa\delta}\int d\sigma d\tau d\lambda \frac{d\bar{x}_\nu}{d\sigma}\frac{d\bar{x}_\kappa}{d\tau}\frac{d\bar{x}_\delta}{d\lambda}\delta^{(4)}\left(x - \bar{x}(\sigma,\tau,\lambda)\right), \tag{11}$$

The monopole gauge transformations (9) express the freedom of distorting the Dirac strings without changing the boundary lines, as can be seen from the transformation

$$\tilde{\delta}_{\lambda\kappa}(x;S_1) \to \tilde{\delta}_{\lambda\kappa}(x;S_2) = \tilde{\delta}_{\lambda\kappa}(x;S_1) + \partial_\mu \delta_\nu(x;V) - \partial_\nu \delta_\mu(x;V), \tag{12}$$

if V is the volume enclosed by the two surfaces S_1 and S_2 with a common boundary line L. For some monopole gauge transformations the string distortions are trivial, namely those of the form $\Lambda^P_\mu = \partial_\mu \Lambda^P$ with $4\Lambda^P = \pi g \sum_{V_4} \delta(x;V_4)$, where $\delta(x;V_4)$ is the δ-function on the four-volume V_4,

$$\delta(x;V_4) \equiv \epsilon_{\mu\nu\kappa\delta}\int d\rho d\sigma d\tau d\lambda \frac{d\bar{x}_\mu}{d\rho}\frac{d\bar{x}_\nu}{d\sigma}\frac{d\bar{x}_\kappa}{d\tau}\frac{d\bar{x}_\delta}{d\lambda}\delta^{(4)}\left(x - \bar{x}(\rho,\sigma,\tau,\lambda)\right). \tag{13}$$

These do not give any change in $F^P_{\mu\nu}$ since they are a submanifold of the original gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$. We may remove them from Λ^P_μ by a gauge-fixing condition such as

$$n_\mu \Lambda^P_\mu \equiv 0, \tag{14}$$

where n_μ is an arbitrary fixed unit vector.
The remaining monopole gauge freedom can be used to bring all Dirac strings to a standard shape so that $F^P_{\mu\nu}(x)$ becomes a function of only the boundary lines L. In fact, with the above n_μ we may always reach the *axial monopole gauge* defined by

$$n_\mu F^P_{\mu\nu} = 0. \tag{15}$$

To see this we take n_μ along the 4-axis and consider the gauge fixing equations

$$F_{4i} + \partial_4 \Lambda^P_i - \partial_i \Lambda^P_4 \;\; = \;\; 0, \quad\;\; i = 1,2,3. \tag{16}$$

With (14) we have $\Lambda^P_4 \equiv 0$ and Λ^P_i could certainly all be determined if they were arbitrary real functions. But the same thing is possible for the restricted class of gauge functions at hand, with the form (10). This is seen most easily by approximating the 4-space by a fine-grained hypercubic lattice of spacing ϵ and imagining $F^P_{\mu\nu}$ to be functions defined on the plaquettes. The δ-functions correspond to integer-valued functions on sites $[\delta(x;V_4)\hat{=}N(x)]$, on links $[\delta_\mu(x;V)\hat{=}N_\mu/\epsilon]$, or plaquettes $[\delta(x;S)\hat{=}N_{\mu\nu}/\epsilon^2]$, and the derivatives ∂_μ to $1/\epsilon$ times lattice differences ∇_μ across links. Thus $F^P_{\mu\nu}$ can be written as $4\pi g N_{\mu\nu}(x)/\epsilon^2$ with integer $N_{\mu\nu}(x)$. The gauge fixing in (16) with the restricted gauge functions amounts then to solving a set of integer-valued equations of the type

$$N_{4i} + \nabla_4 N^P_i - \nabla_i N_4 \;\; = \;\; 0, \quad\;\; i = 1,2,3. \tag{17}$$

with $N_4 \equiv 0$. This is always possible as ha sbeen shown with similar equations in Ref. [4]. With the gauge being fixed we can solve Eq. (8) uniquely by

$$F^P_{\mu\nu}(x) = -8\pi\epsilon_{\mu\nu\lambda\kappa}n_\lambda(n\partial)^{-1}\tilde{\jmath}_\kappa. \tag{18}$$

3) If we want to turn the classical theory associated with (3) into a quantum field theory, we have to take the amplitude $e^{i\mathcal{A}/\hbar}$ and form the path integral over all fluctuating grand-canonical ensembles of world lines L' and worldsheets S. For the world lines it is well known that such a path integral can be replaced by a functional integral over a single fluctuating field [3]. In the absence of monopoles this gives rise, for charged electrons, to the standard quantum field theory of electromagnetism (QED) in which the electric interaction

$$\mathcal{A}_{el} = i\int d^4x A_\mu j_\mu \tag{19}$$

is replaced by the second quantized field action

$$\mathcal{A}_e = \int d^4x \left\{\bar{\psi}_e(x)\left[\gamma^\mu(i\partial_\mu - eA_\mu)\psi_e(x) - m_e\bar{\psi}_e(x)\psi_e(x)\right]\right\}, \tag{20}$$

where m_e is the mass of the electron and $\psi_e(x)$ are the standard Dirac fields of the electron.

4) For the monopoles, the situation is initially much more involved since the path integral is a sum over a grand-canonical ensemble of surfaces S. Up to date, there exists no satisfactory second-quantized field theory which could replace such a sum. The vacuum fluctuations of some non-abelian gauge theory will eventually do the job. Fortunately, however, due to the monopole gauge invariance of the action (3) under (9), most configurations of the surfaces S are physically irrelevant. If we fix the gauge as described above, the monopole gauge field is uniquely given by Eq. (18) and thus depend *only on the orbital worldlines* L of the monopoles via (2). But then

we can rewrite the action with the fixed monopole gauge as

$$\mathcal{A} = \mathcal{A}_1' + \mathcal{A}_{el} + \mathcal{A}_{\lambda\,1} + \mathcal{A}_{\lambda\,2} \qquad (21)$$
$$= \int d^4x \left\{ \left[\frac{1}{16\pi} (F_{\mu\nu} - f^P_{\mu\nu})^2 + iA_\mu j_\mu \right] + i\lambda_{\mu\nu} \left(n_\sigma \partial_\sigma f^P_{\mu\nu} + 8\pi \epsilon_{\mu\nu\lambda\kappa} n_\lambda \tilde{j}_\kappa \right) \right\},$$

where $f^P_{\mu\nu}, \lambda_{\mu\nu}$ are now two *arbitrary* fluctuating fields (i.e., $f^P_{\mu\nu}$ is no longer of the restricted form implied by (6)]. The latter field plays the role of a Lagrange multiplyer to enforce the specific gauge relation (18). The two action terms in which it apears have been denoted by $\mathcal{A}_{\lambda\,1}$ and $\mathcal{A}_{\lambda\,2}$. We have omitted a gauge fixing term for the ordinary electromagnetic gauge since it is standard. The monopole enters now via the magnetic current coupling

$$\mathcal{A}_{mg} \equiv \mathcal{A}_{\lambda\,2} = i \int d^4x \tilde{A}_\kappa \tilde{j}_\kappa, \qquad (22)$$

where $\tilde{A}_\mu$ is short for

$$\tilde{A}_\kappa \equiv 8\pi \lambda_{\mu\nu} \epsilon_{\mu\nu\lambda\kappa} n_\lambda. \qquad (23)$$

The ensemble of monopole orbits L can now be turned into a a single fluctuating field as usual. If monopoles are spin 1/2 particles, this obviously replaces [just as in going from (19) to (20)] the magnetic interaction (22) by

$$\mathcal{A}_g = \int d^4x \left\{ \bar{\psi}_g(x) \gamma^\mu (i\partial_\mu - g\tilde{A}_\mu) \psi_g(x) - m_g \bar{\psi}_g(x) \psi_g(x) \right\}, \qquad (24)$$

where $\psi_g(x)$ is the Dirac field of the monopole. The total gauge-fixed field action is therefore

$$\mathcal{A} = \mathcal{A}_1' + \mathcal{A}_e + \mathcal{A}_g + \mathcal{A}_{\lambda\,1}. \qquad (25)$$

Before gauge fixing, the total path integral for the fluctuating theory involves the fields ψ_e, ψ_g and the gauge fields A_μ and $F^P_{\mu\nu}$. While the path integrals over the first three fields are defined in the standard way as being the product, over all sites x of the above specified hypercubic lattice, of integrals over Grassmann variables $\psi_e(x), \psi_g(x)$ and c-number variables $A_\mu(x)$, the latter is of a new type: It is defined as the product, over all plaquettes of the lattice, of discrete *sums* over all integers $N_{\mu\nu}(x)$ with $F^P_{\mu\nu} = 4\pi g N_{\mu\nu}/\epsilon^2$.

After the gauge fixing, the path integrand will in general contain Fadeev-Popov determinants. For the original gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, any standard gauge fixing, for instance the axial one $\delta(n_\mu A_\mu)$, gives only a trivial constant Fadeev-Popov determinant which can be ignored. The same thing happens with the monopole gauge fixing. At the level of the hypercubic lattice, Kronecker δ's in the path integrand ensure conditions like (16) which corespond to $\prod_{i=1}^3 \delta_{N_{4i},0}$. Also such conditions produce only trivial constant Fadeev-Popov determinants [4], so that there is no need to introduce compensating fermionic ghost fields.

The fluctuations in the gauge-fixed action (25) involves only path integrals over ordinary Grassmann fields ψ_e, ψ_g and c-number fields $A_\mu, f^P_{\mu\nu}, \lambda_{\mu\nu}$. This completes the construction of the quantum field theory of electric charges and Dirac monopoles [5].

Notice that the dependence of this theory on the monopole gauge degree of freedom is much more dramatic than in pure QED. There, it was only a polarization degree of freedom which was made irrelevant by the electromagnetic gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$. Here the monopole gauge transformations (9) reduce the *dimensionality* of the fluctuations from surfaces S to lines L.

5) Let us end by remarking that by integrating out the A_μ field in the classical action (3) we obtain the interaction

$$\mathcal{A}_{int} = \int d^4x \left\{ \frac{1}{16\pi} \left[\left(F^P_{\mu\nu}\right)^2 + 2\partial_\mu F^P_{\mu\nu}(-\partial^2)^{-1}\partial_\lambda F^P_{\lambda\nu}\right)\right] + \frac{4\pi}{2} j_\mu (-\partial^2)^{-1} j_\mu + \frac{1}{2}\partial_\mu F^P_{\mu\nu}(-\partial^2)^{-1} j_\nu \right\}. \quad (26)$$

The third term is the usual electric current-current interaction. The first two terms are seen, via (8), to reduce to the magnetic current-current interaction

$$\mathcal{A}_{\tilde{\jmath}\tilde{\jmath}} = \frac{4\pi}{2} \int d^4x \tilde{\jmath}_\mu (-\partial^2)^{-1} \tilde{\jmath}_\mu. \tag{27}$$

The last term is the current-current interaction between magnetic and electric currents. In the axial gauge with (18) it becomes

$$\mathcal{A}_{j\tilde{\jmath}} = 4\pi \epsilon_{\mu\nu\lambda\kappa} \int d^4x j_\mu (n\partial\partial^2)^{-1} n_\nu \partial_\lambda \tilde{\jmath}_\kappa. \tag{28}$$

These are the correct current-current interactions which can be found in the textbooks [6].

References

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[3] This was first demonstrated by
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For a textbook derivation see
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[4] See Chapter 10 of the second reference in [3].

[5] It is here where our approach shows is superior to the theory with an "exorcized" string of T.T. Wu and C.N. Yang, Phys. Rev. D *14*, 437 (1976), Phys. Rev. D *12*, 3845, (1075), D *16*, 1018 (1977), Nuclear Physics B *107*, 365 (1976), C.N. Yang, Lectures presented at the 1976 Erice summer school, in *Gauge Interactions*, Plenum Press, New York 1978, edited by A. Zichichi and at the 1982 Erice summer school, in *Gauge Interactions*, Plenum Press, New York 1984, edited by A. Zichichi.

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