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Five-loop renormalization group functions of O(n)-symmetric ϕ^4-theory and ϵ-expansions of critical exponents up to ϵ^5

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Motivated by the discovery of errors in six of the 135 diagrams in the published five-loop expansions of the β-function and the anomalous dimensions of the O(n)-symmetric ϕ^4-theory in $D=4-\epsilon$ dimensions we present the results of a full analytic reevaluation of all diagrams. The divergences are removed by minimal subtraction and ϵ-expansions are given for the critical exponents η, ν, and ω up to order ϵ^5.

1. During the last two decades, much effort has been invested into studying the scalar quantum field theory with ϕ^4-interaction. On the one hand, such a theory describes correctly many experimentally observable features of critical phenomena. Field theoretic renormalization group techniques [1] in $D=4-\epsilon$ dimensions [2–4] combined with Borel resummation methods of the resulting ϵ-expansions [5] led to extremely accurate determinations of the critical exponents of all O(n) universality classes. The latter requires the knowledge of the asymptotic behaviour of perturbation series in four dimensions which is completely known in this theory [6]. Apart from such important applications, the ϕ^4-theory, being the simplest renormalizable quantum field theory in the four dimensional space–time, has been an ideal ground for testing new methods of calculating Feynman diagrams and for studying the structure of perturbation theory.

The RG functions of the ϕ^4-theory were first calculated analytically in four dimensions using dimensional regularization [7] and the minimal subtraction (MS) scheme [8] in the three- and four-loop approximations in refs. [9,10]. The critical exponents were obtained as ϵ-expansions [3] up to terms of order ϵ^3 and ϵ^4.

The five-loop anomalous dimension of the field ϕ and the associated critical exponent η to order ϵ^5 were determined analytically in ref. [11]. The five-loop β-function and the anomalous dimension of the mass were given in ref. [12]. However, three of the 124 four-point diagrams contributing to the β-function at the five-loop level could be evaluated only numerically. The analytic calculation of the β-function was finally completed in ref. [13]. The ensuing ϵ-expansions for the critical exponents were obtained up to order ϵ^5 in ref. [14].

Intending further applications, the Berlin group of the authors undertook an independent recalculation of the perturbation series of refs. [11,12], using the same techniques, and discovered errors in six of the 135 diagrams. This meant that the subsequent results of refs. [13,14] were also incorrect. When visiting the Moscow group the errors were confirmed and we can now jointly report all expansions in the correct form.

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2. We consider the $O(n)$-symmetric theory of n real scalar fields ϕ^a ($a=1, 2, ..., n$) with the lagrangian

$$L=\tfrac{1}{2}\,\partial_\mu\phi^a\,\partial_\mu\phi^a+\tfrac{1}{2}m_B^2\phi^a\phi^a+\frac{16\pi^2}{4!}g_B(\phi^a\phi^a)^2\,, \tag{1}$$

in a euclidean space with $D=4-\epsilon$ dimensions. The bare (unrenormalized) coupling constant g_B and mass m_B are expressed via renormalized ones as

$$g_B=\mu^\epsilon Z_g g=\mu^\epsilon\frac{Z_4}{(Z_2)^2}g\,,\quad m_B^2=Z_{m^2}m^2=\frac{Z_{\phi^2}}{Z_2}m^2\,. \tag{2}$$

Here μ is the unit of mass in dimensional regularization and Z_4, Z_2, Z_{m^2} are the renormalization constants of the vertex function, propagator and mass, respectively, with Z_{ϕ^2} being the renormalization constant of the two-point function obtained from the propagator by the insertion, in all possible ways, of the vertex $(-\phi^2)$ [10]. In the MS-scheme the renormalization constants do not depend on dimensional parameters and are expressible as series in $1/\epsilon$ with purely g-dependent coefficients:

$$Z_i=1+\sum_{k=1}^{\infty}\frac{Z_{i,k}(g)}{\epsilon^k}\,. \tag{3}$$

The β-function and the anomalous dimensions entering the RG equations are expressed in the standard way as follows:

$$\beta(g)=\tfrac{1}{2}\epsilon g+\left.\frac{\mathrm{d}g}{\mathrm{d}\ln\mu}\right|_{g_B}=\tfrac{1}{2}g\frac{\partial Z_{g,1}}{\partial g}\,, \tag{4}$$

$$\gamma_m=\left.\frac{\mathrm{d}\ln m}{\mathrm{d}\ln\mu}\right|_{g_B}=-\frac{\mathrm{d}\ln Z_{m^2}}{\mathrm{d}\ln\mu^2}=\tfrac{1}{2}g\frac{\partial Z_{m^2,1}}{\partial g}\,, \tag{5}$$

$$\gamma_i(g)=\left.\frac{\mathrm{d}\ln Z_i}{\mathrm{d}\ln\mu^2}\right|_{g_B}=-\tfrac{1}{2}g\frac{\partial Z_{i,1}}{\partial g}\,,\quad i=2,4,\phi^2\,. \tag{6}$$

We also use the relations

$$\beta(g)=g[2\gamma_2(g)-\gamma_4(g)]\,,\quad \gamma_m(g)=\gamma_2(g)-\gamma_{\phi^2}(g)\,, \tag{7}$$

which follow from the relations between renormalization constants implied by (2) and are useful for the calculations of $\beta(g)$ and $\gamma_m(g)$.

To determine all RG functions up to five loops we calculate the five-loop approximation to the three constants Z_2, Z_4 and Z_{ϕ^2}. The constant Z_2 contains the counterterms of the 11 five-loop propagator diagrams. Two of them were calculated erroneously in ref. [11]. The constant Z_4 receives contributions from 124 vertex diagrams. Of these diagrams, 90 contribute to Z_{ϕ^2} after appropriate changes of combinatorial factors. Four of the 124 counterterms were calculated erroneously in ref. [12].

In the present paper we have used the same methods as in the previous works [11,12] to calculate the counterterms from the dimensionally regularized Feynman integrals, namely, the method of infrared rearrangement [15], the Gegenbauer polynomial x-space technique (GPXT) [16], the integration-by-parts algorithm [17], and the R^*-operation [18]. Three diagrams were calculated analytically first in ref. [13] by using the so-called method of uniqueness, later the same results were obtained for them by using the Gegenbauer polynomials in x-space together with several non-trivial tricks [19]. A detailed description of the calculations including the diagramwise results will be presented in a separate publication.

The analytic results of our recalculation of the five-loop approximations to the RG functions $\beta(g)$, $\gamma_2(g)$ and $\gamma_m(g)$ are [$\zeta(n)$ is the Riemann ζ-function]:

$$\beta(g)=\tfrac{1}{6}g^2(n+8)-\tfrac{1}{6}g^3(3n+14)+\tfrac{1}{432}g^4[33n^2+922n+2960+\zeta(3)\,96(5n+22)]$$
$$-\tfrac{1}{7776}g^5[-5n^3+6320n^2+80456n+196648+\zeta(3)\,96(63n^2+764n+2332)-\zeta(4)\,288(5n+22)(n+8)$$
$$+\zeta(5)\,1920(2n^2+55n+186)]$$
$$+\tfrac{1}{124416}g^6[13n^4+12578n^3+808496n^2+6646336n+13177344$$
$$+\zeta(3)\,16(-9n^4+1248n^3+67640n^2+552280n+1314336)$$
$$+\zeta^2(3)\,768(-6n^3-59n^2+446n+3264)-\zeta(4)\,288(63n^3+1388n^2+9532n+21120)$$
$$+\zeta(5)\,256(305n^3+7466n^2+66986n+165084)-\zeta(6)(n+8)\,9600(2n^2+55n+186)$$
$$-\zeta(7)\,112896(14n^2+189n+526)]\,, \qquad (8)$$

$$\gamma_2(g)=\tfrac{1}{36}g^2(n+2)-\tfrac{1}{432}g^3(n+2)(n+8)+\tfrac{1}{5184}g^4(n+2)[5(-n^2+18n+100)]$$
$$-\tfrac{1}{186624}g^5(n+2)[39n^3+296n^2+22752n+77056-\zeta(3)\,48(n^3-6n^2+64n+184)$$
$$+\zeta(4)\,1152(5n+22)]\,, \qquad (9)$$

$$\gamma_m(g)=\tfrac{1}{6}g(n+2)-\tfrac{1}{36}g^2(n+2)[5]+\tfrac{1}{72}g^3(n+2)[5n+37]$$
$$-\tfrac{1}{15552}g^4(n+2)[-n^2+7578n+31060+\zeta(3)\,48(3n^2+10n+68)+\zeta(4)\,288(5n+22)]$$
$$+\tfrac{1}{373248}g^5(n+2)[21n^3+45254n^2+1077120n+3166528+\zeta(3)\,48(17n^3+940n^2+8208n+31848)$$
$$-\zeta^2(3)\,768(2n^2+145n+582)+\zeta(4)\,288(-3n^3+29n^2+816n+2668)$$
$$+\zeta(5)\,768(-5n^2+14n+72)+\zeta(6)\,9600(2n^2+55n+186)]\,. \qquad (10)$$

For $n=1$ the series have the numerical form

$$\beta(g)=1.5g^2-2.833g^3+16.27g^4-135.8g^5+1424.2841g^6\,, \qquad (11)$$

$$\gamma_2=0.0833g^2-0.0625g^3+0.3385g^4-1.9256g^5\,, \qquad (12)$$

$$\gamma_m=0.5g-0.4167g^2+1.75g^3-9.978g^4+75.3778g^5\,. \qquad (13)$$

Note that the five-loop coefficients have changed by about 0.3% for the β-function, by about 9% for γ_m, and by a factor of three for γ_2 in comparison with the wrong results of refs. [11,12].

3. These RG functions can now be used to calculate the critical exponents describing the behaviour of a statistical system near the critical point of the second order phase transition [4]. At the critical temperature $T=T_C$, the asymptotic behaviour of the correlation function for $|\boldsymbol{x}|\to\infty$ has the form

$$\Gamma(\boldsymbol{x})\sim\frac{1}{|\boldsymbol{x}|^{D-2+\eta}}\,. \qquad (14)$$

Close to T_C, the correlation length behaves for $t=T-T_C\to 0$ as

$$\xi\sim t^{-\nu}(1+\text{const.}\cdot t^{\omega\nu}+...)\,. \qquad (15)$$

The three critical exponents η, ν and ω defined in this way completely specify the critical behaviour of the system. All other exponents can be expressed in terms of these [4].

The three critical exponents can be determined from the RG functions of the ϕ^4-theory by going to the infrared-stable fixed point

$$g=g_0(\epsilon)=\sum_{k=1}^{\infty} g^{(k)}\epsilon^k \,, \tag{16}$$

which is determined by the condition ($\beta_\epsilon \equiv \beta - \frac{1}{2}\epsilon g$)

$$\beta_\epsilon'(g_0)=0\,,\quad \beta_\epsilon(g_0)=[\partial\beta_\epsilon(g)/\partial g]_{g=g_0}>0\,. \tag{17}$$

The resulting formulas for the critical exponents are

$$\eta=2\gamma_2(g_0)\,,\quad 1/\nu=2[1-\gamma_m(g_0)]\,,\quad w=2\beta_\epsilon'(g_0)\,, \tag{18}$$

each emerging as an ϵ-expansion up to order ϵ^5. From (8)–(10) we therefore find

$$\eta(\epsilon)=\frac{(n+2)\epsilon^2}{2(n+8)^2}\bigg(1+\frac{\epsilon}{4(n+8)^2}(-n^2+56n+272)$$
$$-\frac{\epsilon^2}{16(n+8)^4}[5n^4+230n^3-1124n^2-17920n-46144+\zeta(3)(n+8)\,384(5n+22)]$$
$$-\frac{\epsilon^3}{64(n+8)^6}[13n^6+946n^5+27620n^4+121472n^3-262528n^2-2912768n-5655552$$
$$-\zeta(3)(n+8)\,16(n^5+10n^4+1220n^3-1136n^2-68672n-171264)$$
$$+\zeta(4)(n+8)^3\,1152(5n+22)-\zeta(5)(n+8)^2\,5120(2n^2+55n+186)]\bigg)\,, \tag{19}$$

$$1/\nu(\epsilon)=2+\frac{(n+2)\epsilon}{n+8}\bigg(-1-\frac{\epsilon}{2(n+8)^2}(13n+44)$$
$$+\frac{\epsilon^2}{8(n+8)^4}[3n^3-452n^2-2672n-5312+\zeta(3)(n+8)\,96(5n+22)]$$
$$+\frac{\epsilon^3}{32(n+8)^6}[3n^5+398n^4-12900n^3-81552n^2-219968n-357120$$
$$+\zeta(3)(n+8)\,16(3n^4-194n^3+148n^2+9472n+19488)$$
$$+\zeta(4)(n+8)^3\,288(5n+22)-\zeta(5)(n+8)^2\,1280(2n^2+55n+186)]$$
$$+\frac{\epsilon^4}{128(n+8)^8}[3n^7-1198n^6-27484n^5-1055344n^4-5242112n^3-5256704n^2+6999040n-626688$$
$$-\zeta(3)(n+8)\,16(13n^6-310n^5+19004n^4+102400n^3-381536n^2-2792576n-4240640)$$
$$-\zeta^2(3)(n+8)^2\,1024(2n^4+18n^3+981n^2+6994n+11688)$$
$$+\zeta(4)(n+8)^3\,48(3n^4-194n^3+148n^2+9472n+19488)$$
$$+\zeta(5)(n+8)^2\,256(155n^4+3026n^3+989n^2-66018n-130608)$$
$$-\zeta(6)(n+8)^4\,6400(2n^2+55n+186)+\zeta(7)(n+8)^3\,56448(14n^2+189n+526)]\bigg)\,, \tag{20}$$

$$\omega(\epsilon)=\epsilon-\frac{\epsilon^2}{(n+8)^2}(9n+42)+\frac{\epsilon^3}{4(n+8)^4}[33n^3+538n^2+4288n+9568+\zeta(3)(n+8)\,96(5n+22)]$$
$$+\frac{\epsilon^4}{16(n+8)^6}[5n^5-1488n^4-46616n^3-419528n^2-1750080n-2599552$$
$$-\zeta(3)(n+8)\,96(63n^3+548n^2+1916n+3872)$$
$$+\zeta(4)(n+8)^3\,288(5n+22)-\zeta(5)(n+8)^2\,1920(2n^2+55n+186)]$$
$$+\frac{\epsilon^5}{64(n+8)^8}[13n^7+7196n^6+240328n^5+3760776n^4+38877056n^3+223778048n^2+660389888n$$
$$+752420864$$
$$-\zeta(3)(n+8)\,16(9n^6-1104n^5-11648n^4-243864n^3-2413248n^2-9603328n-14734080)$$
$$-\zeta^2(3)(n+8)^2\,768(6n^4+107n^3+1826n^2+9008n+8736)$$
$$-\zeta(4)(n+8)^3\,288(63n^3+548n^2+1916n+3872)$$
$$+\zeta(5)(n+8)^2\,256(305n^4+7386n^3+45654n^2+143212n+226992)$$
$$-\zeta(6)(n+8)^4\,9600(2n^2+55n+186)+\zeta(7)(n+8)^3\,112896(14n^2+189n+526)]\,. \quad (21)$$

For $n=1$, these expansions read, numerically,

$$\eta=0.01852\epsilon^2+0.01869\epsilon^3-0.00833\epsilon^4+0.02566\epsilon^5\,, \quad (22)$$

$$\frac{1}{\nu}=2-0.333\epsilon-0.1173\epsilon^2+0.1245\epsilon^3-0.307\epsilon^4+0.951\epsilon^5\,, \quad (23)$$

$$\omega=\epsilon-0.630\epsilon^2+1.618\epsilon^3-5.24\epsilon^4+20.75\epsilon^5\,. \quad (24)$$

Note that $\eta^{(5)}$ has decreased by about 30% in comparison with the (incorrect) results of ref. [14], $\nu^{(5)}$ has increased by about 10%, and $\omega^{(5)}$ increased by about 0.6% in comparison with ref. [14].

It is known that the series of the ϵ-expansion are asymptotic series and special resummation techniques [20,21] should be applied to obtain reliable estimates of the critical exponents. Although the size of the ϵ^5 terms in the physical dimension (i.e., at $\epsilon=1$) is very large, their contribution to the exponents in the *resummed* series is very small. This is why even large relative changes of the ϵ^5 coefficients turn out not to change the critical exponents [#1] within the accuracy of previous determinations [14,22].

[#1] This was checked in detail by Janke and Kleinert using the resummation programs of ref. [21]. Since the deviations from the previous numbers were found to be very small the results were not published.

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