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Relations between Markov Processes via Local Time and Coordinate Transformations

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The Duru-Kleinert method of solving unknown path integrals of quantum mechanical systems by relating them to known ones does not apply to Markov processes since the Duru-Kleinert transform of a Fokker-Planck equation is, in general, not a Fokker-Planck equation. In this Letter, we present a significant modification of the method, based again on a combination of path-dependent time and coordinate transformations, to obtain such relations after all. As an application we express unknown Green functions for a one-parameter family of Markov processes in terms of the known one for the Schenzle-Brand process. [S0031-9007(96)02010-8]

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The stochastic theory of Markov processes [1,2] explains many phenomena where fluctuations play a significant role. Prominent examples are provided by the emergence of self-organization [3], the occurrence of quantum dissipation [4], and the appearance of stochastic resonance [5]. In accordance with their wide range of applicability there exist various powerful solution methods for Markov processes based either on the global characterization of the probability evolution by the Onsager-Machlup path integral or on its local equivalent, the Fokker-Planck equation. Examples for the former are the small noise expansion and the adiabatic elimination procedure of fast random variables, for the latter the eigenfunction expansion and the continued-fraction method applied to periodically driven systems [6].

In this Letter we demonstrate the use of path-dependent time transformations, which have become a powerful tool for solving quantum mechanical problems since Duru and Kleinert's original work on the path integral of the hydrogen atom [7,8]. In one dimension with coordinate q, the crucial time transformation has the form

$$\frac{dt}{ds} = f(q)\,, \tag{1}$$

where $f(q)$ is some positive but otherwise arbitrary function. Such a transformation does not change the standard formulations of quantum mechanics, a form invariance which has recently been emphasized in [9,10]. Because of this form invariance, different quantum mechanical systems can be related to each other.

Some years ago, the Duru-Kleinert (DK) method was also applied to the stochastic theory of Markov processes [11,12], thereby relating the Fokker-Planck equation of Markov processes to other stochastic differential equations. The latter had, however, an important disadvantage: They were no longer Fokker-Planck equations so that the DK transformations did not link different Markov processes. This defect will be eliminated in the sequel by a significant modification of the DK method.

Consider a one-dimensional Markov process of a single random variable x, whose conditional probability density $P(x, x_0; t)$ possesses the initial condition

$$P(x, x_0; 0) = \delta(x - x_0) \tag{2}$$

and obeys the Fokker-Planck equation for $t > 0$,

$$\partial_t P(x, x_0; t) = \hat{H}(x) P(x, x_0; t)\,. \tag{3}$$

Here $\hat{H}(x)$ denotes the infinitesimal time evolution operator,

$$\hat{H}(x)\bullet = -\frac{\partial}{\partial x}[K(x)\bullet] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[D(x)\bullet], \tag{4}$$

containing the drift coefficient $K(x)$ and the diffusion coefficient $D(x)$. Then the Laplace transform

$$G(x, x_0; E) = \int_0^\infty dt\, e^{-Et} P(x, x_0; t) \tag{5}$$

represents a fixed-energy Green function solving the time-independent equation

$$[\hat{H}^{(\mathrm{i})}(x) - E^{(\mathrm{i})}]G^{(\mathrm{i})}(x, x_0; E^{(\mathrm{i})}) + \delta(x - x_0) = 0. \quad (6)$$

We have added a superscript (i) to emphasize that this equation describes the initial stochastic system from which we depart.

It is well known [6] that the Fokker-Planck equation remains form invariant under arbitrary invertible coordinate transformations

$$x = x(q), \quad (7)$$

where the Green function transforms as

$$G^{(1)}(q, q_0; E^{(\mathrm{i})}) = \pm x'(q) G^{(\mathrm{i})}\big(x(q), x(q_0); E^{(\mathrm{i})}\big). \quad (8)$$

The different signs take into account whether (7) is monotonously increasing or decreasing. In fact, we conclude from (4) and (6) that $G^{(1)}(q, q_0; E^{(\mathrm{i})})$ satisfies the Fokker-Planck equation

$$[\hat{H}^{(1)}(q) - E^{(\mathrm{i})}]G^{(1)}(q, q_0; E^{(\mathrm{i})}) + \delta(q - q_0) = 0, \quad (9)$$

where the new infinitesimal time evolution operator

$$\hat{H}^{(1)}(q)\bullet = -\frac{\partial}{\partial q}[K^{(1)}(q)\bullet] + \frac{1}{2}\frac{\partial^2}{\partial q^2}[D^{(1)}(q)\bullet] \quad (10)$$

contains the drift and diffusion coefficients

$$K^{(1)}(q) = \frac{1}{x'(q)} K^{(\mathrm{i})}\big(x(q)\big) - \frac{x''(q)}{2x'^3(q)} D^{(\mathrm{i})}\big(x(q)\big), \quad (11)$$

$$D^{(1)}(q) = \frac{1}{x'^2(q)} D^{(\mathrm{i})}\big(x(q)\big). \quad (12)$$

These coordinate transformations are of standard use in finding unknown solutions from known ones [13,14].

Let us now supplement these transformations by the path-dependent time transformation (1). First, we proceed in analogy with Ref. [8], Chap. 12, and change the Green function according to

$$G^{(1)}(q, q_0; E^{(\mathrm{i})}) = f(q)\frac{F(q_0; E^{(\mathrm{i})})}{F(q; E^{(\mathrm{i})})} \times G^{(\mathrm{f})}\big(q, q_0; E^{(\mathrm{f})}(E^{(\mathrm{i})})\big), \quad (13)$$

where $F(q; E^{(\mathrm{i})})$ and $E^{(\mathrm{f})}(E^{(\mathrm{i})})$ are as yet unknown trial functions. Applying (9) with (10), we find the equation for the final Green function

$$[\hat{H}^{(\mathrm{f})}(q) - E^{(\mathrm{f})}(E^{(\mathrm{i})}) + X(q; E^{(\mathrm{i})})] \times G^{(\mathrm{f})}\big(q, q_0; E^{(\mathrm{f})}(E^{(\mathrm{i})})\big) + \delta(q - q_0) = 0, \quad (14)$$

with the infinitesimal time evolution operator

$$\hat{H}^{(\mathrm{f})}(q)\bullet = -\frac{\partial}{\partial q}[K^{(\mathrm{f})}(q)\bullet] + \frac{1}{2}\frac{\partial^2}{\partial q^2}[D^{(\mathrm{f})}(q)\bullet], \quad (15)$$

containing the transformed drift and diffusion coefficients

$$K^{(\mathrm{f})}(q) = f(q)\left[K^{(1)}(q) + \frac{F'(q; E^{(\mathrm{i})})}{F(q; E^{(\mathrm{i})})} D^{(1)}(q)\right], \quad (16)$$

$$D^{(\mathrm{f})}(q) = f(q) D^{(1)}(q), \quad (17)$$

and the additional term

$$X(q; E^{(\mathrm{i})}) = f(q)\Bigg[\frac{1}{2} D^{(1)}(q)\frac{F''(q; E^{(\mathrm{i})})}{F(q; E^{(\mathrm{i})})} + K^{(1)}(q)\frac{F'(q; E^{(\mathrm{i})})}{F(q; E^{(\mathrm{i})})} + \mathcal{E}(q; E^{(\mathrm{i})}) - E^{(\mathrm{i})}\Bigg], \quad (18)$$

where

$$\mathcal{E}(q; E^{(\mathrm{i})}) = E^{(\mathrm{f})}(E^{(\mathrm{i})})/f(q). \quad (19)$$

Equation (14) has the above-mentioned defect of not being a Fokker-Planck equation, due to the presence of the additional term $X(q; E^{(\mathrm{i})})$. This term can, however, be removed by choosing any functions $E^{(\mathrm{f})}(E^{(\mathrm{i})})$ and $F(q; E^{(\mathrm{i})})$ which solve the differential equation

$$X(q; E^{(\mathrm{i})}) \equiv 0. \quad (20)$$

Note that although this equation is of the same complexity as the initial Fokker-Planck equation, only a *particular solution* is required, so that labor will definitely be saved by our method.

An alternative procedure which avoids solving the differential equation (20) is by leaving the time transformation function $f(q)$ open, choosing some trial function $F(q; E^{(\mathrm{i})})$, and calculating $\mathcal{E}(q; E^{(\mathrm{i})})$ from (18) with (20). If this happens to be factorizable as in (19), the q-dependent prefactor may be chosen as the transformation function $f(q)$ and the $E^{(\mathrm{i})}$-dependent one as the energy function $E^{(\mathrm{f})}(E^{(\mathrm{i})})$.

In either procedure, the function $F(q; E^{(\mathrm{i})})$ is subject to an important restriction. In the limit $E^{(\mathrm{i})} \to 0$ it has to satisfy

$$\lim_{E^{(\mathrm{i})} \to 0} F(q; E^{(\mathrm{i})}) = 1 \quad (21)$$

identically in q, so that the energy function $E^{(\mathrm{f})}(E^{(\mathrm{i})})$ obeys

$$\lim_{E^{(\mathrm{i})} \to 0} E^{(\mathrm{f})}(E^{(\mathrm{i})}) = 0. \quad (22)$$

Only under this condition do initial and final Green functions possess proper stationary limits

$$p_{\mathrm{st}}^{(\mathrm{i})}(x) = \lim_{E^{(\mathrm{i})} \to 0} E^{(\mathrm{i})} G^{(\mathrm{i})}(x, x_0; E^{(\mathrm{i})}), \quad (23)$$

$$p_{\mathrm{st}}^{(\mathrm{f})}(q) = \lim_{E^{(\mathrm{f})} \to 0} E^{(\mathrm{f})} G^{(\mathrm{f})}(q, q_0; E^{(\mathrm{f})}). \quad (24)$$

From (8), (13), and (21)–(24), we read off a relation between them

$$p_{\mathrm{st}}^{(\mathrm{i})}(x) = \pm\left[\frac{dE^{(\mathrm{f})}(E^{(\mathrm{i})})}{dE^{(\mathrm{i})}}\right]^{-1}_{E^{(\mathrm{i})}=0} \frac{f\big(q(x)\big)}{x'\big(q(x)\big)} p_{\mathrm{st}}^{(\mathrm{f})}\big(q(x)\big), \quad (25)$$

which guarantees the normalization of the pobability:

$$\int p_{\rm st}^{(\rm i)}(x)\,dx = \int p_{\rm st}^{(\rm f)}(q)\,dq = 1\,. \tag{26}$$

An interesting feature of the present method is that it permits us in the stationary limit to relate the probability distributions of two *arbitrary* Markov processes to each other. Given initial and final drift and diffusion coefficients we satisfy (12) and (17) by choosing the time transformation function as

$$f(q) = x'^2(q)D^{(\rm f)}(q)/D^{(\rm i)}(x(q))\,. \tag{27}$$

Using this together with (11), (12), (16), (17), and (21), we obtain the desired coordinate transformation from the differential equation

$$x'(q) = C\exp\Big[\int^{x(q)} d\tilde{x}\,\frac{2K^{(\rm i)}(\tilde{x})}{D^{(\rm i)}(\tilde{x})} - \int^{q} d\tilde{q}\,\frac{2K^{(\rm f)}(\tilde{q})}{D^{(\rm f)}(\tilde{q})}\Big], \tag{28}$$

where C is an integration constant. The final stationary solution

$$p_{\rm st}^{(\rm f)}(q) = \frac{N^{(\rm f)}}{D^{(\rm f)}(q)}\,\exp\Big[\int^{q} d\tilde{q}\,\frac{2K^{(\rm f)}(\overline{q})}{D^{(\rm f)}(\tilde{q})}\Big] \tag{29}$$

is then related to the initial one

$$p_{\rm st}^{(\rm i)}(x) = \frac{N^{(\rm i)}}{D^{(\rm i)}(x)}\,\exp\Big[\int^{x} d\tilde{x}\,\frac{2K^{(\rm i)}(\tilde{x})}{D^{(\rm i)}(\tilde{x})}\Big] \tag{30}$$

by (25), (27), and (28), if the normalization constants satisfy

$$C = \frac{N^{(\rm i)}}{N^{(\rm f)}}\,\frac{dE^{(\rm f)}(E^{(\rm i)})}{dE^{(\rm i)}}\bigg|_{E^{(\rm i)}=0}. \tag{31}$$

In order to demonstrate the applicability of the new transformation method, we consider a Markov process with a multiplicative noise for a random variable $x \in (0,\infty)$, where the drift and the diffusion coefficient depend on an arbitrary parameter $\alpha > 0$ as follows:

$$K^{(\rm i)}(x) = a^{(\rm i)}x - b^{(\rm i)}x^{2\alpha+1}\,, \qquad D^{(\rm i)}(x) = Q^{(\rm i)}x^{2\alpha+2}. \tag{32}$$

Performing a transformation of the random variable (7) and the time (1) with

$$x(q) = q^{\beta}, \qquad f(q) = q^{\gamma}, \tag{33}$$

the associated differential equation (20) with (18) is solved by a function

$$F(q;E^{(\rm i)}) = q^{\delta(E^{(\rm i)})}, \tag{34}$$

if the parameters β and γ are related according to

$$2\alpha\beta + \gamma = 0\,. \tag{35}$$

The function $\delta(E^{(\rm i)})$ and the energy relation $E^{(\rm f)} = E^{(\rm f)}(E^{(\rm i)})$ are determined by

$$\delta(E^{(\rm i)}) = (\beta/a^{(\rm i)})E^{(\rm i)}, \tag{36}$$

$$E^{(\rm f)}(E^{(\rm i)}) = -\frac{Q^{(\rm i)}}{2a^{(\rm i)2}}\,E^{(\rm i)2} + \Big[\frac{b^{(\rm i)}}{a^{(\rm i)}} + \frac{Q^{(\rm i)}}{2a^{(\rm i)}}\Big]E^{(\rm i)}. \tag{37}$$

Note that (34), (36), and (37) satisfy the correct limits (21) and (22).

The transformed drift and diffusion coefficients then follow from (11), (12), (16), and (17) as

$$K^{(\rm f)}(q) = a^{(\rm f)}q - b^{(\rm f)}q^{-2\alpha\beta+1}, \qquad D^{(\rm f)}(q) = Q^{(\rm f)}q^2, \tag{38}$$

where

$$a^{(\rm f)} = -\frac{b^{(\rm i)}}{\beta} + \frac{Q^{(\rm i)}}{\beta}\Big[\frac{E^{(\rm i)}}{a^{(\rm i)}} - \frac{\beta-1}{2\beta}\Big], \tag{39}$$

$$b^{(\rm f)} = -a^{(\rm i)}/\beta, \qquad Q^{(\rm f)} = Q^{(\rm i)}/\beta^2. \tag{40}$$

These relations supply us with solutions of the Fokker-Planck equation for a one-parameter family of Markov processes (32) if we specialize

$$\beta = -1/\alpha\,. \tag{41}$$

Then the final Markov process (38) for a random variable $q \in (0,\infty)$ coincides with the well-understood Schenzle-Brand process [14], which is a standard model in nonlinear optics and chemical reaction dynamics. It can be derived as an approximation to a number of different processes by adiabatically eliminating fast random variables in the limit of large external fluctuations. For instance, the Schenzle-Brand process describes the electrical field near a laser threshold, the multiplicative noise being due to inversion fluctuations.

With standard methods [6], the Fokker-Planck equation of the Schenzle-Brand process with (38) and (41) can be transformed to the Schrödinger equation of the Morse oscillator. As the quantum mechanical Green function of this system has been explicitly calculated from path integrals [15,16], the Green function of the Schenzle-Brand process is known:

$$G^{(\rm f)}(q,q_0;E^{(\rm f)}) = \frac{\Gamma(\frac{k^{(\rm f)}}{2} + \frac{1}{4} - \frac{a^{(\rm f)}}{2Q^{(\rm f)}})}{b^{(\rm f)}\Gamma(1+k^{(\rm f)})}\,q^{a^{(\rm f)}/Q^{(\rm f)}-5/2}q_0^{-(a^{(\rm f)}/Q^{(\rm f)})-1/2}\exp\Big[\frac{b^{(\rm f)}(q_0^2-q^2)}{2Q^{(\rm f)}}\Big]$$
$$\times\Big\{\Theta(q-q_0)W_{1/4+a^{(\rm f)}/2Q^{(\rm f)},k^{(\rm f)}/2}\Big(\frac{b^{(\rm f)}q^2}{Q^{(\rm f)}}\Big)M_{1/4+a^{(\rm f)}/2Q^{(\rm f)},k^{(\rm f)}/2}\Big(\frac{b^{(\rm f)}q_0^2}{Q^{(\rm f)}}\Big) + (q\leftrightarrow q_0)\Big\}, \tag{42}$$

where $k^{(\mathrm{f})}$ denotes the abbreviation

$$k^{(\mathrm{f})} = \sqrt{(a^{(\mathrm{f})}/Q^{(\mathrm{f})} - 1/2)^2 + 2E^{(\mathrm{f})}/Q^{(\mathrm{f})}}. \quad (43)$$

With the help of the transformation formulas (8) and (13), and taking into account (33)–(41), we find from this the unknown Green function of the initial Markov processes (32):

$$G^{(\mathrm{i})}(x, x_0; E^{(\mathrm{i})}) = \frac{\Gamma(\frac{E^{(\mathrm{i})}}{2\alpha a^{(\mathrm{i})}})}{a^{(\mathrm{i})}\Gamma(1 + k^{(\mathrm{i})})} x^{-\alpha k^{(\mathrm{i})} - \alpha - 1} x_0^{\alpha k^{(\mathrm{i})} + \alpha} \exp\left[\frac{a^{(\mathrm{i})}}{2\alpha Q^{(\mathrm{i})}}\left(\frac{1}{x_0^{2\alpha}} - \frac{1}{x^{2\alpha}}\right)\right]$$
$$\times \left\{\Theta(x_0 - x) W_{k^{(\mathrm{i})}/2+1/2-E^{(\mathrm{i})}/2\alpha a^{(\mathrm{i})}, k^{(\mathrm{i})}/2}\left(\frac{a^{(\mathrm{i})}}{\alpha Q^{(\mathrm{i})} x^{2\alpha}}\right) M_{k^{(\mathrm{i})}/2+1/2-E^{(\mathrm{i})}/2\alpha a^{(\mathrm{i})}, k^{(\mathrm{i})}/2}\left(\frac{a^{(\mathrm{i})}}{\alpha Q^{(\mathrm{i})} x_0^{2\alpha}}\right) + (x \leftrightarrow x_0)\right\}, \quad (44)$$

with

$$k^{(\mathrm{i})} = b^{(\mathrm{i})}/\alpha Q^{(\mathrm{i})} + 1/2\alpha. \quad (45)$$

Furthermore, we obtain from (24), (42), and (43) the stationary solution of the Schenzle-Brand process (38) and (41)

$$p_{\mathrm{st}}^{(\mathrm{f})}(q) = \frac{2(\frac{b^{(\mathrm{f})}}{Q^{(\mathrm{f})}})^{a^{(\mathrm{f})}/Q^{(\mathrm{f})} - 1/2}}{\Gamma(\frac{a^{(\mathrm{f})}}{Q^{(\mathrm{f})}} - \frac{1}{2})} q^{2a^{(\mathrm{f})}/Q^{(\mathrm{f})} - 2} \exp\left[-\frac{b^{(\mathrm{f})}}{Q^{(\mathrm{f})}} q^2\right], \quad (46)$$

which is mapped via (25) and (33)–(41) to the initial Markov process (32), yielding

$$p_{\mathrm{st}}^{(\mathrm{i})}(x) = \frac{2\alpha(\frac{a^{(\mathrm{i})}}{\alpha Q^{(\mathrm{i})}})^{b^{(\mathrm{i})}/\alpha Q^{(\mathrm{i})} + 1/2\alpha + 1}}{\Gamma(\frac{b^{(\mathrm{i})}}{\alpha Q^{(\mathrm{i})}} + \frac{1}{2\alpha} + 1)}$$
$$\times x^{-2(b^{(\mathrm{i})}/Q^{(\mathrm{i})}) - 2\alpha - 2} \exp\left[-\frac{a^{(\mathrm{i})}}{\alpha Q^{(\mathrm{i})} x^{2\alpha}}\right]. \quad (47)$$

The analytic properties of the Green functions $G^{(\mathrm{i})}(x, x_0; E^{(\mathrm{i})})$ and $G^{(\mathrm{f})}(q, q_0; E^{(\mathrm{f})})$ in the energies $E^{(\mathrm{i})}$ and $E^{(\mathrm{f})}$ determine the spectra of the infinitesimal time evolution operators $\hat{H}^{(\mathrm{i})}(x)$ and $\hat{H}^{(\mathrm{f})}(q)$ (Ref. [8], Chap. 9). From (42)–(45) we deduce that the initial multiplicative process (32) has only a discrete spectrum, whereas the final one, (38) and (41), contains both a discrete and a continuous branch. Such differences between spectral types were encountered before in quantum mechanical DK transformations: The hydrogen atom has discrete and continuous states, whereas the DK-equivalent oscillator has only discrete states [7,8]. The spectra are usually related by a Sommerfeld-Watson transformation of the Green functions (see [17] and Ref. [8], Chap. 14).

The discrete levels closest to zero are in a one-to-one correspondence [9,18]. For stochastic systems, these levels rule the approach of the conditional probability density to its stationary limit. In our example, the associated poles of the initial Green function (44) and (45)

$$E_n^{(\mathrm{i})} = -2\alpha a^{(\mathrm{i})} n, \qquad n = 0, 1, \ldots \quad (48)$$

are mapped to the corresponding final ones of (42) and (43)

$$E_n^{(\mathrm{f})} = 2Q^{(\mathrm{f})} n^2 + (Q^{(\mathrm{f})} - 2a^{(\mathrm{f})}) n \quad (49)$$

by

$$E_n^{(\mathrm{f})} = E^{(\mathrm{f})}(E_n^{(\mathrm{i})}), \quad (50)$$

as long as n is bounded by

$$n \leq a^{(\mathrm{f})}/2Q^{(\mathrm{f})} - 1/4. \quad (51)$$

The evaluation of (51) requires the use of relations (37) and (39)–(41).

In summary, we have shown that a combination of local time and coordinate transformations opens new possibilities of relating different Markov processes. By extending the method to several random variables, we expect many useful applications. Furthermore, we hope that this method might help to solve non-Markov processes [19].

Let us finally mention that the local time transformation (1) is nonholonomic in space-time, i.e., it carries a flat space-time into a space-time with nonzero torsion and curvature [9,10]. If we allow for purely spatial nonholonomic changes of coordinates, we can also reach space-time geometries with spatial curvature and torsion. This will enable us to describe technically relevant diffusion processes in crystals with defects [20–23].

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