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Strong-coupling ϕ^4 -theory in $4 - \epsilon$ dimensions, and critical exponents

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Abstract

With the help of variational perturbation theory we continue the renormalization constants of ϕ^4 -theories in $4 - \epsilon$ dimensions to infinitely strong bare couplings g_0 and find their power behavior in g_0 , thereby determining all critical exponents without the standard renormalization group techniques. © 1998 Published by Elsevier Science B.V. All rights reserved.

1. In a recent paper [1] we have shown that there exists a simple way of extracting the strong-coupling properties of a ϕ^4 -theory from perturbation expansions. In particular, we were able to find the power behavior of the renormalization constants in the limit of large couplings, and from this all critical exponents of the system. By using the known expansion coefficients of the renormalization constants in three dimensions up to six loops we derived extremely accurate values for the critical exponents. The method is a systematic extension of the Feynman-Kleinert variational approximation to path integrals [2] to arbitrary orders [3]. For an anharmonic oscillator, the derived variational perturbation expansions converge uniformly and exponentially fast, like $e^{-\text{const} \times N^{1/3}}$ in the order N of the approximation [4,5]. The same type of convergence seems to exist also for the

ϕ^4 -theory if the power $1/3$ is replaced by $1 - \omega$, where ω is the critical exponent governing the approach to scaling [1]. This exponent plays a crucial role in the development of the theory.

2. Variational perturbation expansions have the important property of possessing a good strong-coupling limit, as was first shown for the harmonic oscillator [6,7]. The speed of convergence turned out to be governed by the convergence radius of the strong-coupling expansion [8,9]. The good strong-coupling properties have enabled us to set up a simple algorithm for deriving uniformly convergent approximations to functions of which one knows a few Taylor coefficients and an important scaling property: they approach a constant value with a given inverse power of the variable. The renormalized coupling constant g of a ϕ^4 -theory has precisely this property as a function of the bare coupling constant g_0 . In $D = 4 - \epsilon$ dimensions, it approaches

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a constant value g^* for increasing bare coupling constant g_0 like

$$g(g_0) = g^* - \frac{\text{const}}{g_0^{\omega/\epsilon}} + \dots, \quad (1)$$

where g^* is the infrared-stable fixed point and ω is called the critical exponent of the approach to scaling. The same exponent governs the approach to scaling of every function of g which behaves like $f(g) = f(g^*) + f'(g^*) \times \text{const}/g_0^{\omega/\epsilon} + \dots$.

The purpose of this paper is to point out that the theory developed in [1] for a three-dimensional ϕ^4 -theory can easily be applied in $D = 4 - \epsilon$ dimensions with beautiful results at the two-loop level.

3. Let us briefly recall the relevant formulas. Consider a function $f(g_0)$ for which we know the first $N + 1$ expansion terms, $f_N(g_0) = \sum_{n=0}^N a_n g_0^n$, and the fact that it approaches a constant value f^* in the form of an inverse power series $f_M(g_0) = \sum_{m=0}^M b_m (g_0^{-2/q})^m$ with a finite convergence radius g_s (simple examples were treated in [10]). Then the N th approximation to the value f^* is obtained from the formula

$$f_N^* = \text{opt}_{\hat{g}_0} \left[\sum_{j=0}^N a_{j0} \hat{g}_0^j \sum_{k=0}^{N-j} \binom{-qj/2}{k} (-1)^k \right], \quad (2)$$

where the expression in brackets has to be optimized in the variational parameter \hat{g}_0 . The optimum is the smoothest among all real extrema. If there are no real extrema, the turning points serve the same purpose.

The derivation of this expression is simple: We replace g_0 in $f_N(g_0)$ trivially by $\bar{g}_0 \equiv g_0/\kappa^q$ with $\kappa = 1$. Then we rewrite, again trivially, κ^{-q} as $(K^2 + \kappa^2 - K^2)^{-q/2}$ with an arbitrary parameter K . Each term is now expanded in powers of $r = (\kappa^2 - K^2)/K^2$ assuming r to be of the order g_0 . Taking the limit $g_0 \rightarrow \infty$ at a fixed ratio $\hat{g}_0 \equiv g_0/K^q$, so that $K \rightarrow \infty$ like $g_0^{1/q}$ and $r \rightarrow -1$, we obtain (2). Since the final result to all orders cannot depend on the arbitrary parameter K , we expect the best result to any finite order to be optimal at an extremal value of K , i.e., of \hat{g}_0 .

The strong-coupling approach to the limiting value $r = -1 + \kappa^2/K^2 = -1 + O(g_0^{-2/q})$ implies the leading correction to f_N^* to be of the order of $g_0^{-2/q}$.

Application of the theory to a function with the strong-coupling behavior (1) requires therefore a parameter $q = 2\epsilon/\omega$ in formula (2).

For $N = 2$ and 3 one can give analytic expressions for the strong-coupling limits (2). Setting $\rho \equiv 1 + q/2 = 1 + \epsilon/\omega$, we find for $N = 2$

$$f_2^* = \text{opt}_{\hat{g}_0} [a_0 + a_1 \rho \hat{g}_0 + a_2 \hat{g}_0^2] = a_0 - \frac{1}{4} \frac{a_1^2}{a_2} \rho^2. \quad (3)$$

For $N = 3$, we obtain from the extrema

$$\begin{aligned} f_3^* &= \text{opt}_{\hat{g}_0} \left[a_0 + \frac{1}{2} a_1 \rho (\rho + 1) \hat{g}_0 \right. \\ &\quad \left. + a_2 (2\rho - 1) \hat{g}_0^2 + a_3 \hat{g}_0^3 \right] \\ &= a_0 - \frac{1}{3} \frac{\bar{a}_1 \bar{a}_2}{a_3} \left(1 - \frac{2}{3} r\right) + \frac{2}{27} \frac{\bar{a}_2^3}{a_3^2} (1 - r), \quad (4) \end{aligned}$$

where $r \equiv \sqrt{1 - 3\bar{a}_1 a_3 / \bar{a}_2^2}$ and $\bar{a}_1 \equiv \frac{1}{2} a_1 \rho (\rho + 1)$ and $\bar{a}_2 \equiv a_2 (2\rho - 1)$. The positive square root must be taken to connect g_3^* smoothly to g_2^* in the limit of a vanishing coefficient of g_0^3 . If the square root is imaginary, the optimum is given by the unique turning point, leading once more to (4) but with $r = 0$.

The parameter $\rho = 1 + \epsilon/\omega$ can be determined from the expansion coefficients of a function $F(g_0)$ as follows. Assuming $F(g_0)$ to be constant F^* in the strong-coupling limit, the logarithmic derivative $f(g_0) \equiv g_0 F'(g_0)/F(g_0)$ must vanish at $g_0 = \infty$. If $F(g_0)$ starts out as $A_0 + A_1 g_0 + \dots$ or $A_1 g_0 + A_2 g_0^2 + \dots$, the logarithmic derivative is

$$\begin{aligned} f(g_0) &= A'_1 g_0 + (2A'_2 - A_1'^2) g_0^2 \\ &\quad + (A_1'^3 - 3A'_1 A'_2 + 3A_3') g_0^3 + \dots, \quad (5) \end{aligned}$$

where $A'_i = A_i/A_0$, or

$$\begin{aligned} f(g_0) &= 1 + \hat{A}_2 g_0 + (2\hat{A}_3 - \hat{A}_2^2) g_0^2 \\ &\quad + (\hat{A}_2^3 - 3\hat{A}_2 \hat{A}_3 + 3\hat{A}_4) g_0^3 + \dots, \quad (6) \end{aligned}$$

where $\hat{A}_i = A_i/A_1$. The expansion coefficients on the right-hand sides are then inserted into (3) or (4), and the left-hand sides have to vanish to ensure that $F(g_0) \rightarrow F^*$.

If the approach $F(g_0) \rightarrow F^*$ is of the type (1), the function

$$h(g_0) \equiv g_0 \frac{F''(g_0)}{F'(g_0)} = 2\hat{A}_2 g_0 + (-4\hat{A}_2^2 + 6\hat{A}_3) g_0^2 + (8\hat{A}_2^3 - 18\hat{A}_2\hat{A}_3 + 12\hat{A}_4) g_0^3 + \dots \quad (7)$$

must have the strong-coupling limit

$$h(g_0) \rightarrow h^* = -\frac{\omega}{\epsilon} - 1. \quad (8)$$

4. These formulas are now applied to the renormalization constants of the ϕ^4 -theory in $D = 4 - \epsilon$ dimensions with the bare euclidean action

$$\mathcal{A} = \int d^D x \left\{ \frac{1}{2} [\partial\phi_0(x)]^2 + \frac{1}{2} m_0^2 \phi_0^2(x) + (4\pi)^2 \frac{\lambda_0}{4!} [\phi_0^2(x)]^2 \right\}. \quad (9)$$

The field $\phi_0(x)$ is an n -dimensional vector, and the action is $O(n)$ -symmetric in this vector space. The Ising model corresponds to $n = 1$, the critical behavior of percolation is described by $n = 0$, superfluid phase transitions by $n = 2$, and classical Heisenberg magnetic systems by $n = 3$.

By calculating the Feynman integrals regularized via an expansion in $\epsilon = 4 - D$ with the help of an arbitrary mass scale μ , one obtains renormalized values of mass, coupling constant, and field related to the bare input quantities by renormalization constants Z_ϕ, Z_m, Z_g :

$$m_0^2 = m^2 Z_m Z_\phi^{-1}, \quad \lambda_0 = \lambda Z_g Z_\phi^{-2}, \quad \phi_0 = \phi Z_\phi^{1/2}. \quad (10)$$

Up to two loops, perturbation theory yields the following expansions in powers of the dimensionless reduced coupling constant $g_0 \equiv \lambda_0/\mu^\epsilon$:

$$g = g_0 - \frac{n+8}{3\epsilon} g_0^2 + \left[\frac{(n+8)^2}{9\epsilon^2} + \frac{3n+14}{6\epsilon} \right] g_0^3, \quad (11)$$

$$\frac{m^2}{m_0^2} = 1 - \frac{n+2}{3} \frac{g_0}{\epsilon} + \frac{n+2}{9} \left[\frac{n+5}{\epsilon^2} + \frac{5}{4\epsilon} \right] g_0^2, \quad (12)$$

$$\frac{\phi^2}{\phi_0^2} = 1 + \frac{n+2}{36} \frac{g_0^2}{\epsilon}. \quad (13)$$

We now set the scale parameter μ equal to m and consider all quantities as functions of $g_0 = \lambda_0/m^\epsilon$. In order to describe second-order phase transitions, we let m_0^2 go to zero like $\tau = \text{const} \times (T - T_c)$ as the temperature T approaches the critical temperature T_c , and assume that also m^2 goes to zero, and thus g_0 to infinity. The latter assumption will be shown to be self-consistent after Eq. (28)

$$\frac{m^2}{m_0^2} \propto g_0^{-\eta_m/\epsilon} \propto m^{\eta_m}, \quad \frac{\phi^2}{\phi_0^2} \propto g_0^{\eta/\epsilon} \propto m^{-\eta}. \quad (14)$$

The powers can therefore be calculated from the strong-coupling limits of the logarithmic derivatives

$$\eta_m(g_0) = -\epsilon \frac{d}{d \log g_0} \log \frac{m^2}{m_0^2}, \quad \eta(g_0) = \epsilon \frac{d}{d \log g_0} \log \frac{\phi^2}{\phi_0^2}. \quad (15)$$

Inserting (12) and (13) yields the expansions

$$\eta_m(g_0) = \frac{n+2}{3} g_0 - \frac{n+2}{18} \left(5 + 2 \frac{n+8}{\epsilon} \right) g_0^2, \quad (16)$$

$$\eta(g_0) = \frac{n+2}{18} g_0^2. \quad (17)$$

When approaching the second-order phase transitions, where the bare mass m_0^2 vanishes like $\tau \equiv (T - T_c)$, the physical mass m^2 vanishes with a different power of τ . This power is obtained from the first equation in (14), which shows that $m \propto \tau^{1/(2-\eta_m)}$. Experiments observe that the coherence length of fluctuations $\xi = 1/m$ increases near T_c like $\tau^{-\nu}$, so that we derive for the critical exponent ν a value $1/(2-\eta_m)$. Similarly we see from the first equation in (14) that the scaling dimension $D/2 - 1$ of the free field ϕ_0 for $T \rightarrow T_c$ is changed, in the strong-coupling limit $g_0 \rightarrow \infty$, to $D/2 - 1 + \eta/2$, the number η being the so-called anomalous dimension of the field. This implies a change in the large-distance behavior of the correlation functions $\langle \phi(x)\phi(0) \rangle$ at T_c from the free-field behavior r^{-D+2} to $r^{-D+2-\eta}$. The magnetic susceptibility is determined by the integrated correlation function

$\langle \phi_0(x)\phi_0(0) \rangle$. At zero coupling constant g_0 , this is proportional to $1/m_0^2 \propto \tau^{-1}$, which is changed by fluctuations to $m^{-2}\phi_0^2/\phi^2$. This has a temperature behavior $m^{-(2-\eta)} = \tau^{-\nu(2-\eta)} \equiv \tau^{-\gamma}$, which defines the critical exponent $\gamma = \nu(2 - \eta)$ observable in magnetic experiments. Using $\nu = 1/(2 - \eta_m)$ and the expansions (16), (17), we obtain for $\gamma(g_0)$ the perturbation expansion up to second order in g_0 :

$$\gamma(g_0) = 1 + \frac{n+2}{6}g_0 + \frac{n+2}{36}\left(n-4-2\frac{n+8}{\epsilon}\right)g_0^2. \quad (18)$$

This is certainly positive, so that the first equation (Eq. (14)) ensures that with m_0^2 also m^2 goes to zero, a necessary condition for the self-consistency of our theory.

All calculations in this note will be restricted to the two-loop level, which will be sufficient to demonstrate the power and beauty of the new strong-coupling theory with analytical results.

5. We begin by calculating the critical exponent ω from the requirement that $g(g_0)$ has a constant strong-coupling limit, implying the vanishing of (6) for $g_0 \rightarrow \infty$. From the expansion (11) we obtain a logarithmic derivative (6) up to the term g_0^2 , so that Eq. (3) can be used to find the scaling condition

$$0 = 1 - \frac{1}{4} \frac{\hat{A}_2^2}{2\hat{A}_3 - \hat{A}_2^2} \rho^2. \quad (19)$$

This gives

$$\rho = \sqrt{8\hat{A}_3/\hat{A}_2^2 - 4}. \quad (20)$$

Since ω must be greater than zero, only the positive square root is physical. With the explicit coefficients A_1, A_2, A_3 of expansion (11), this becomes

$$\rho = 2\sqrt{1 + 3\frac{3n+14}{(n+8)^2}\epsilon}. \quad (21)$$

The associated critical exponent $\omega = \epsilon/(\rho - 1)$ is plotted in Fig. 1. It has the ϵ -expansion

$$\omega = \epsilon - 3\frac{3n+14}{(n+8)^2}\epsilon^2 + \dots, \quad (22)$$

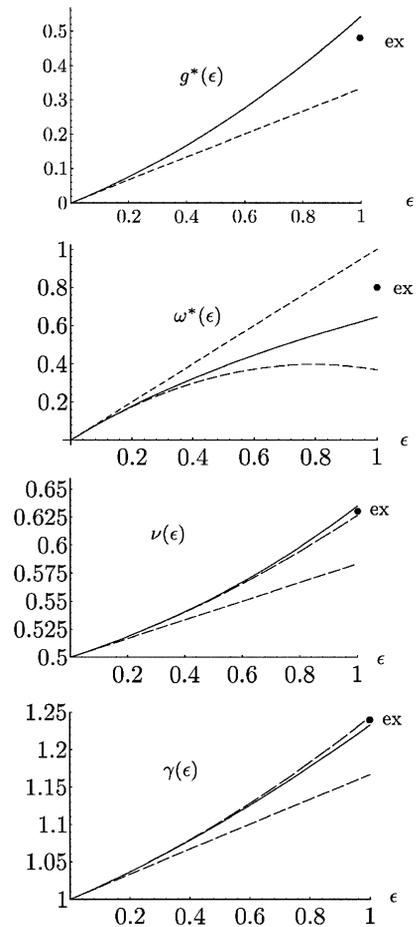


Fig. 1. For the Ising universality class ($n=1$), the first figure shows the renormalized coupling at infinite bare coupling as a function of $\epsilon = 4 - D$ calculated via variational perturbation theory from the first two perturbative expansion terms. The curve coincides with the ϵ -expansion up to order ϵ^2 . The dashed curve indicates the linear term. The other figures show the critical exponents ω , ν , and γ . Dashed curves indicate linear and quadratic ϵ -expansions. The dots mark presently accepted values of $g^* \approx 0.48 \pm 0.003$, $\omega \approx 0.802 \pm 0.003$, $\nu = 0.630 \pm 0.002$, and $\gamma = 1.241 \pm 0.004$ obtained from six-loop calculations [1].

which is also shown in Fig. 1, and agrees with the first two terms obtained from renormalization group calculations [11].

From Eqs. (8), (7), and (3) we obtain for the critical exponent ω a further equation

$$-\frac{\omega}{\epsilon} - 1 = -\frac{\rho}{\rho - 1} = -\frac{1}{2} \frac{\hat{A}_2^2 \rho^2}{3\hat{A}_3 - 2\hat{A}_2^2}, \quad (23)$$

which is solved by

$$\rho = \frac{1}{2} + \sqrt{\frac{6\hat{A}_3}{\hat{A}_2^2} - \frac{15}{4}}, \quad (24)$$

with the positive sign of the square root ensuring a positive ω . Inserting the coefficients of (11), this becomes

$$\rho = \frac{1}{2} + \frac{3}{2} \sqrt{1 + 4 \frac{3n+14}{(n+8)^2} \epsilon}. \quad (25)$$

The associated critical exponent $\omega = \epsilon/(\rho - 1)$ has the same ϵ -expansion (22) as the previous approximation (21). The full approximations based on (25) is indistinguishable from the earlier one in the plot of Fig. 1.

Having determined ω , we can now calculate g^* . Inserting the first two coefficients of the expansion (11) into (3) we obtain

$$g_2^* = a_0 - \frac{1}{4} \frac{a_1^2}{a_2} \rho^2. \quad (26)$$

Inserting (21), this yields

$$g_2^* = \frac{3}{n+8} \epsilon + 9 \frac{3n+14}{(n+8)^3} \epsilon^2, \quad (27)$$

which is precisely the well-known ϵ -expansion of g^* in renormalization group calculations up to the second order. Including the next coefficient, we can use formula (4) to calculate the next approximation g_3^* . At $\epsilon = 1$, the square root turns out to be imaginary, so that it has to be omitted (corresponding to the turning point as optimum). The resulting curve lies slightly ($\approx 8\%$) above the curve (27), i.e., represents a worse approximation than (27). Indeed, the ϵ^3 -term in g_3^* is $81(3n+14)^2/8(n+8)^5$ and disagrees in sign with the exact term $\epsilon^3 [3(-33n^3 + 110n^2 + 1760n + 4544)/8 - 36\zeta(3)(n+8)(5n+22)]/(n+8)^5$, which we would find by calculating ρ from an expansion (11) with one more power in g_0 .

We now turn to the critical exponent ν . Taking the expansion (16) to $g_0 \rightarrow \infty$, we obtain from formula (3) the limiting value

$$\eta_m = \frac{\epsilon}{4} \frac{n+2}{n+8+5\epsilon/2} \rho^2. \quad (28)$$

The corresponding $\nu = 1/(2 - \eta_m)$ is plotted in Fig. 1. With the approximation (21) for ρ we find for ν the ϵ -expansion

$$\nu = \frac{1}{2} + \frac{1}{4} \frac{n+2}{n+8} \epsilon + \frac{(n+2)(n+3)(n+20)}{8(n+8)^3} \epsilon^2 + \dots, \quad (29)$$

which is also shown in Fig. 1, and agrees with renormalization group results to this order.

As a third independent critical exponent we calculate $\gamma = (2 - \eta)/(2 - \eta_m)$ by inserting the coefficients of the expansion (18) into formula (3), which yields

$$\gamma = 1 + \frac{\epsilon}{8} \frac{n+2}{n+8 - (n-4)\epsilon/2} \rho^2, \quad (30)$$

plotted in Fig. 1. This has an ϵ -expansion

$$\gamma = 1 + \frac{1}{2} \frac{n+2}{n+8} \epsilon + \frac{1}{4} \frac{(n+2)(n^2 + 22n + 52)}{(n+8)^3} \epsilon^2 + \dots, \quad (31)$$

shown again in Fig. 1, and agreeing with renormalization group results to this order. The full approximation is plotted in Fig. 1. The critical exponent $\eta = 2 - \gamma/\nu$ has the ϵ -expansion $\eta = (n+2)\epsilon^2/2(n+8)^2 + \dots$.

6. We conclude that variational strong-coupling theory can easily be applied to ϕ^4 -theories in $D = 4 - \epsilon$ dimensions and yields resummed expressions for the ϵ -dependence of all critical exponents. Their ϵ -expansions agree with those obtained from renormalization group calculations. The calculations here are supposed to illustrate the new calculational procedure and do not yet give very accurate results, in particular for the critical exponent ω whose expansion coefficients have the worst growth at large orders. In order to achieve better accuracies than presently available, we shall go to five loops and incorporate knowledge of the large-order behavior of the expansion coefficients.

Let us finally mention that while we have studied here the approach to the critical properties of the theory in the disordered phase where $m_0^2 > 0$, the

asymptotic behavior (1) with $g_0 = \lambda_0/m^\epsilon$ yields course the same limit g^* and critical exponents ω, ν, η, \dots if the critical point is approached in the ordered phase where $m_0^2 < 0$, the opposite sign of m_0^2 producing merely a phase factor in the next-to-leading term.

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