Two phase transitions in chiral Gross-Neveu model in $2 + \epsilon$ dimensions at low N

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Abstract

We show that the chiral Gross-Neveu model in $2 + \epsilon$ dimensions has for a small number N of fermions two phase transitions corresponding to *pair formation* and *pair condensation*. In the first transition, fermions and antifermions acquire spontaneously a mass and are bound to pairs which behave like a Bose liquid in a chirally symmetric state. In the second transition, the Bose liquid condenses into a coherent state which breaks chiral symmetry. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

The Nambu-Jona-Lasino (NJL) model [1] and its N-component version, the Gross-Neveu (GN) model [2], are field theories of zero-mass fermions with quartic interaction which provide us with considerable insight into the mechanisms of spontaneous symmetry breakdown. Both models can formally be turned into pure boson theories. In an $SU(3) \times SU(3)$-symmetric version, the NJL model has been shown to be equivalent to a chirally $SU(3) \times SU(3)$ invariant σ-model which reproduces all well-known relations of current algebra [3]. For recent work and citations see [4].

The Gross-Neveu model is exactly solvable in the limit of $N \to \infty$. For an attractive sign of the interaction, a collective fermion-antifermion field acquires a nonzero vacuum expectation, and the system shows quasi-long-range order. In $D = 2 + \epsilon$ dimensions, the order of this state becomes proper long-range. The ordered state is reached in a second-order phase transition from a disordered state if the renormalized coupling constant g becomes larger that a critical value $g^* = \pi\epsilon$. The disordered state at small $g < g^*$ consists of massless interacting fermions. It exhibits chiral symmetry, in which fermions are transformed by a phase rotation containing a γ_5 matrix. In the ordered state at larger $g > g^*$, however, the fermions acquire spontaneously a mass, and the chiral symmetry is broken spontaneously.

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The purpose of this note is to point out, that in a modified Gross-Neveu Model in which pairs of fermions form bound pair states analogous to the Cooper pairs in superconductor, the paired phase becomes incoherent in a Kosterlitz-Thouless-like transition [5] if the number of field components N drops below a certain critical value $N_c \lesssim 8$. From this we conclude that in $2 + \epsilon$ dimensions, the transition in which the pairs form exists independently of a transition in which they condense.

In the ordinary Gross-Neveu model, the role of the Kosterlitz-Thouless-like transition is played by an Ising transition, which appears in addition to the transition in which the collective state forms.

2. Proper Gross-Neveu model

The original Gross-Neveu model has the following $O(N)$-symmetric Lagrange density

$$\mathscr{L} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g_0}{2N} \left(\bar{\psi}_a \psi_a \right)^2, \tag{1}$$

where the index a runs from 1 to N. At the mean-field level, the effective action is equal to the initial action

$$\Gamma[\Psi, \bar{\Psi}] = \mathscr{A}[\Psi, \bar{\Psi}] = \int d^D x \left[\bar{\Psi} i \partial\!\!\!/ \Psi + \frac{g_0}{2N} \left(\bar{\Psi}_a \Psi_a \right)^2 \right]. \tag{2}$$

In general, we obtain all Green functions from the generating functional

$$Z[\eta, \bar{\eta}] = e^{iW[\eta, \bar{\eta}]} = \int \mathscr{D}\psi \mathscr{D}\bar{\psi} \exp\left\{ i\mathscr{A}[\psi, \bar{\psi}] + i(\bar{\psi}\eta + \text{c.c.}) \right\}, \tag{3}$$

where $\eta(x)$ and $\bar{\eta}(x)$ are fermionic anticommuting sources. A collective field $\sigma \sim g\bar{\psi}\psi$ is introduced to rewrite (3) as

$$Z[\eta, \bar{\eta}] = \int \mathscr{D}\psi \mathscr{D}\bar{\psi} \mathscr{D}\sigma \exp\left\{ i \int d^D x \left[\bar{\psi}_a (i\partial\!\!\!/ - \sigma) \psi_a + (\bar{\psi}\eta + \text{c.c.}) - N\sigma^2 / 2g_0 \right] \right\}. \tag{4}$$

The fields $\psi(x)$ are integrated out according to the rule to yield a generating functional containing only the collective field $\sigma(x)$:

$$Z[\eta, \bar{\eta}] = \int \mathscr{D}\sigma \exp\{ i\mathscr{A}_{\text{coll}}[\sigma] - \bar{\eta}G_\sigma \eta \}, \tag{5}$$

with the collective action

$$\mathscr{A}_{\text{coll}}[\sigma] = N\left\{ -\frac{1}{2g_0}\sigma^2 - i\text{Tr}\log[i\partial\!\!\!/ - \sigma(x)] \right\}. \tag{6}$$

where Tr denotes functional and Dirac trace.

In the limit $N \to \infty$, the field σ is squeezed into the extremum of the action and we obtain the effective action

$$\frac{1}{N}\Gamma[\Sigma, \psi, \bar{\psi}] = -\frac{1}{2g_0}\Sigma^2(x) - i\text{Tr}\log[i\partial\!\!\!/ - \Sigma(x)] + \frac{1}{N}\bar{\Psi}_a[i\partial\!\!\!/ - \Sigma(x)]\Psi_a \tag{7}$$

The extremum of $\Gamma[\Sigma, \psi, \bar{\psi}]$ is given by the equations of motion,

$$[i\partial\!\!\!/ - \Sigma(x)]\Psi_a(x) = 0, \tag{8}$$

$$\Sigma(x) = g_0 \text{tr}\, G_\Sigma(x,x) - \frac{1}{N}g_0 \bar{\psi}_a \psi_a(x), \quad G_\Sigma(x,y) = \frac{i}{i\partial\!\!\!/ - \Sigma}, \tag{9}$$

where the trace symbol tr is restricted to the Dirac indices. The expectation Ψ_a of a fermionic field is always zero, so that we only must solve the *gap equation*

$$\Sigma(x) = g_0 \mathrm{tr}\, G_\Sigma(x,x).\tag{10}$$

Thus, as far as the extremum is concerned, we may study only the purely collective part of the exact action

$$\frac{1}{N}\Gamma[\Sigma] = -\frac{1}{2g_0}\Sigma^2 - i\mathrm{Tr}\log iG_\Sigma^{-1}.\tag{11}$$

The ground state is given by a constant gap field Σ_0, for which (10) yields either $\Sigma_0 = 0$ or

$$1 = g_0 \,\mathrm{tr}(1)\int \frac{d^D p_E}{(2\pi)^D}\,\frac{1}{p_E^2 + \Sigma_0^2},\tag{12}$$

where we have performed a Wick rotation $p^0 \to ip^4$ to euclidean momenta $p_\mu \equiv (p^1, p^2, p^3, p^4)$ with the metric $p_E^2 = -p^2$. The Dirac matrices have dropped out, except for the unit matrix whose trace is $2^{D/2}$ for even D. This expression may be continued to any non-integer value of D.

For a constant Σ, the effective action gives rise to an effective potential

$$\frac{1}{N}v(\Sigma) = -\frac{1}{N}\Gamma[\Sigma] = \frac{1}{2g_0}\Sigma^2 - \mathrm{tr}(1)\tfrac{1}{2}\int \frac{d^D p_E}{(2\pi)^D}\log\big[\,p_E^2 + \Sigma^2\,\big].\tag{13}$$

Performing the integral yields in $D = 2 + \epsilon$ dimensions with $\epsilon > 0$

$$\frac{1}{N}v(\Sigma) = \frac{\mu^\epsilon}{2}\left[\frac{\Sigma^2}{g_0\,\mu^\epsilon} - b_\epsilon\left(\frac{\Sigma}{\mu}\right)^{2+\epsilon}\mu^2\right],\tag{14}$$

where μ is an arbitrary mass scale, and the constant b_ϵ stands for

$$b_\epsilon = \frac{2}{D}2^{\epsilon/2}\bar{S}_D\,\Gamma(D/2)\,\Gamma(1-D/2) = \frac{2}{D}\frac{1}{(2\pi)^{D/2}}\Gamma(1-D/2),\tag{15}$$

which has an ϵ-expansion $b_\epsilon \sim -\big[1 - (\epsilon/2)\log(2\pi e^{-\gamma})\big]/\pi\epsilon + \mathscr{O}(\epsilon)$. A renormalized coupling constant g may be introduced by the equation

$$\frac{1}{g_0\,\mu^\epsilon} - b_\epsilon \equiv \frac{1}{g},\tag{16}$$

so that

$$\frac{1}{N}v(\Sigma) = \frac{\mu^\epsilon}{2}\left\{\frac{\Sigma^2}{g} + b_\epsilon\Sigma^2\left[1 - \left(\frac{\Sigma}{\mu}\right)^\epsilon\right]\right\}.\tag{17}$$

Extremizing this we obtain either $\Sigma_0 = 0$ or a nonzero Σ_0 solving the gap equation (12) in the form

$$1 - \frac{g^*}{g} = \frac{D}{2}\left(\frac{\Sigma_0}{\mu}\right)^\epsilon,\tag{18}$$

where $g^* = -1/b_\epsilon \approx \pi\epsilon$. A nontrivial solution of this is called *gap*. It specifies the mass which the fermions acquire from the attractive interactions, and will be denoted by M. The second derivative of $v(\Sigma)$ shows that

the solutions $\Sigma_0 = 0$ and $\Sigma_0 \neq 0$ are stable for $g_0 > 0$ and $g_0 < 0$, respectively. Denoting the solution Σ_0 of (18) for $g = \infty$ by M_∞, we may write the g-dependence of the gap as

$$M(g) = M_\infty \left(1 - \frac{g^*}{g} \right)^{1/\epsilon}. \tag{19}$$

In terms of M, the effective potential (17) can be rewritten as

$$\frac{1}{N} v(\Sigma) = \frac{b_\epsilon}{4} M^D \left[D \left(\frac{\Sigma}{M} \right)^2 - 2 \left(\frac{\Sigma}{M} \right)^D \right]. \tag{20}$$

It has a minimum at $\Sigma = M$, where it yields the condensation energy $v(M) = -NM^D/4\pi$.

3. Correlation functions of pair field

If N is no longer infinite, the pair field σ in the partition function (4) performs fluctuations around the extremal value $\Sigma_0 = M$. For large N, the correlation functions of $\sigma(x)$ can be extracted from the leading effective action (7) at $\Sigma_0 = M$. Setting $\Sigma(x) = M + \Sigma'(x)$, we expand

$$\delta^2 \Gamma = -\frac{N}{2} \left[\frac{\Sigma'^2}{g_0} + i \, \text{Tr} \left(\frac{i}{i\slashed{\partial} - M} \Sigma' \frac{i}{i\slashed{\partial} - M} \Sigma' \right) \right], \tag{21}$$

implying a propagator of the σ'-field in momentum space

$$G_{\sigma'\sigma'} = -\frac{1}{N} \frac{i}{1/g_0 + \Pi(q)}, \tag{22}$$

where $\Pi(q)$ is given by the self-energy diagram

$$\Pi(q) = -2^{D/2} \int \frac{d^D k_E}{(2\pi)^D} \frac{k(k-q)_E - M^2}{(k_E^2 + M^2) \left[(k-q)_E^2 + M^2 \right]},$$

th integral being performed over euclidean D-dimensional energy-momenta. With standard Feynman methods, this can be transformed into a simple integral

$$\Pi(q) = -\frac{D(D-1)}{2} b_\epsilon M^\epsilon \int_0^1 dx \left[\frac{q_E^2}{M^2} x(1-x) + 1 \right]^{\epsilon/2}. \tag{23}$$

Inserting here Eq. (18), we obtain

$$\frac{1}{g_0} + \Pi(q) = \mu^\epsilon \left(\frac{1}{g^*} - \frac{1}{g} \right) \left\{ -1 + (D-1) \int_0^1 dx \left[\frac{q_E^2}{M^2} x(1-x) + 1 \right]^{\epsilon/2} \right\}, \tag{24}$$

which can be expanded for small q as

$$\frac{1}{g_0} + \Pi(q) = \epsilon \mu^\epsilon \left(\frac{1}{g^*} - \frac{1}{g} \right) \left[1 + (D-1) \frac{1}{12} \frac{q_E^2}{M^2} + \dots \right]. \tag{25}$$

This shows that the propagator (22) has a correlation length

$$\xi = \left(\frac{D-1}{12 M^2} \right)^{1/2}. \tag{26}$$

Inserting the g-dependence of M from (19), we see that

$$\xi = \frac{1}{M_\infty}\left(\frac{D-1}{12}\right)^{1/2}\left(1 - \frac{g^*}{g}\right)^{-1/\epsilon} \tag{27}$$

so that the coherence length diverges for $g \to g^*$ with a critical exponent $\nu = 1/\epsilon$.

3.1. Chiral and complex pair field version of model with Goldstone bosons

We want to prove the existence of two phase transitions in the chiral version of the Gross-Neveu model, whose Lagrange density is

$$\mathscr{L} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g_0}{2N}\left[\left(\bar{\psi}_a \psi_a\right)^2 + \left(\bar{\psi}_a i \gamma_5 \psi_a\right)^2\right]. \tag{28}$$

The collective field action (6) is then replaced by

$$\mathscr{A}_{\text{coll}}[\sigma] = N\left\{-\frac{1}{2g_0}(\sigma^2 + \pi^2) - i\text{Tr}\log\left[i\partial\!\!\!/ - \sigma(x) - i\gamma_5\pi\right]\right\}. \tag{29}$$

This model is invariant under the continuous set of chiral O(2) transformations which rotate σ and π fields into each other. This model is equivalent to yet another one which is closely related to the BCS model of superconductivity. Its Lagrange density is

$$\mathscr{L} = \bar{\psi}_a i \partial\!\!\!/ \psi_a + \frac{g_0}{2N}\left(\bar{\psi}_a C\bar{\psi}_a^T\right)\left(\psi_b^T C\psi_b\right). \tag{30}$$

Here C is the matrix of charge conjugation which is defined by

$$C\gamma^\mu C^{-1} = -\gamma^{\mu T}. \tag{31}$$

In two dimensions, we choose the γ-matrices as $\gamma^0 = \sigma^1$, $\gamma^1 = -i\sigma^2$, and $C = \gamma^1$. Note that $\left(\bar{\psi}_a C\bar{\psi}_a^T\right)^\dagger = -\psi_a^T C\psi_a$, implying that $g_0 < 0$ corresponds to an attractive potential. The second model goes over into the first by replacing $\psi \to \frac{1}{2}(1 - \gamma_5)\psi + \frac{1}{2}(1 + \gamma_5)C\bar{\psi}^T$, where superscript T denotes transposition. In the Lagrange density (30) we introduce a complex collective field by adding a term $-(N/2g_0)\left|\Delta - \frac{g_0}{N}\psi_b^T C\psi_b\right|^2$, leading to the partition function

$$Z[\eta,\bar{\eta}] = \int \mathscr{D}\psi\mathscr{D}\bar{\psi}\mathscr{D}\Delta\mathscr{D}\Delta^\dagger \exp\left\{i\int d^D x\left[\bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{1}{2}\left(\Delta^\dagger\psi_a^T C\psi_a + \text{c.c.}\right) + \bar{\psi}\eta + \bar{\eta}\psi - \frac{N}{2g_0}|\Delta|^2\right]\right\}. \tag{32}$$

The relation with the previous collective fields σ and π is $\Delta = \sigma + i\pi$. In order to integrate out the Fermi fields we rewrite the free part of Lagrange density in the matrix form

$$\frac{1}{2}\left(\psi^T C, \bar{\psi}\right)\begin{pmatrix} 0 & i\partial\!\!\!/ \\ i\partial\!\!\!/ & 0 \end{pmatrix}\begin{pmatrix} \psi \\ C\bar{\psi}^T \end{pmatrix} \tag{33}$$

which is the same as $\bar{\psi}i\partial\!\!\!/\psi$, since $\psi^T CC\bar{\psi}^T = \bar{\psi}\psi$, $\psi^T C\overleftrightarrow{\partial\!\!\!/}C\bar{\psi}^T = \bar{\psi}\overleftrightarrow{\partial\!\!\!/}\psi$. But then the interaction with Δ can be combined with (33) in the form $\frac{1}{2}\phi_i^T G_\Delta^{-1}\phi$, where

$$\phi = \begin{pmatrix} \psi \\ C\bar{\psi}^T \end{pmatrix}, \quad \phi^T = \left(\psi^T, \bar{\psi}C^{-1}\right) \tag{34}$$

are doubled fermion fields, and

$$iG_\Delta^{-1} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} \Delta^\dagger & i\partial\!\!\!/ \\ i\partial\!\!\!/ & \Delta \end{pmatrix} = -\left(iG_\Delta^{-1}\right)^T \tag{35}$$

is the inverse propagator in the presence of the external field Δ. Now we perform the functional integral over the fermion fields, and obtain

$$Z[j] = \int \mathcal{D}\Delta\mathcal{D}\Delta^\dagger e^{iN\mathcal{A}[\Delta] + \frac{1}{2}j_a^T G_\Delta j_a}, \tag{36}$$

where $\mathcal{A}[\Delta]$ is the collective action

$$\mathcal{A}[\Delta] = -\tfrac{1}{2g_0}\int d^D x |\Delta|^2 - \frac{i}{2}\operatorname{Tr}\log iG_\Delta^{-1} \tag{37}$$

and j_a is the doubled version of the external source

$$j = \begin{pmatrix} \bar{\eta}^T \\ C^{-1}\eta \end{pmatrix}. \tag{38}$$

This is chosen so that $\bar{\psi}\eta + \bar{\eta}\psi = \frac{1}{2}(j^T\phi - \phi^T j)$. In the limit $N \to \infty$, we obtain from (36) the effective action

$$\frac{1}{N}\Gamma[\Delta,\Psi] = -\frac{1}{2g_0}|\Delta|^2 - \frac{i}{2}\operatorname{Tr}\log iG_\Delta^{-1} + \frac{1}{N}\bar{\Psi}_a iG_\Delta^{-1}\Psi_a \tag{39}$$

in the same way as in the last chapter for the simpler model with a real σ-field.

The ground state has $\Psi = 0$, so that the minimum of the effective action implies for Δ_0 either $\Delta_0 = 0$ or the gap equation

$$1 = -g_0 \operatorname{Tr}\left[G_{\Delta_0}\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}\right] \tag{40}$$

where we may assume Δ_0 to be real. With the Green function

$$G_{\Delta_0}(x,y) = \int \frac{d^D p}{(2\pi)^D} e^{-ip(x-y)} \frac{i}{p^2 - \Delta_0}\begin{pmatrix} -\Delta_0 & p\!\!\!/ \\ p\!\!\!/ & \Delta_0 \end{pmatrix}\begin{pmatrix} C^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix}, \tag{41}$$

the gap equation (40) takes the same form as (12):

$$1 = g_0 \operatorname{tr}(1)\int \frac{d^D p}{(2\pi)^D}\frac{1}{p^2 + M^2}, \tag{42}$$

where we have again set $M \equiv \Delta_0$. The renormalization of the coupling constant and of the effective potential yields the same equation for $v(\Delta_0) = v(M)$ as before, so that the previous stability discussion for $g < g^*$ and $g > g^*$ holds also here.

Let us now study the propagator of the complete Δ-field. For small deviations $\Delta' \equiv \Delta - \Delta_0$ away from the ground state value we find from (39) the quadratic term

$$\frac{1}{N}\delta^2\Gamma = -\frac{1}{2}\left\{\frac{|\Delta'|^2}{g_0} + \frac{i}{2}\operatorname{Tr}\left[\begin{pmatrix} \Delta'^\dagger & \\ & \Delta' \end{pmatrix}G_M\begin{pmatrix} \Delta'^\dagger & \\ & \Delta' \end{pmatrix}G_M\right]\right\}.$$

The second term in curly brackets may be written more explicitly as an integral $\int d^D q/(2\pi)^D$ over

$$\frac{i}{2}\left[M^2(\Delta'^2 + \Delta'^{*2})2^{D/2}\int\frac{d^D k}{(2\pi)^D}\frac{i}{k^2 - M^2}\frac{i}{(k-q)^2 - M^2}\right.$$

$$\left.+2|\Delta'|^2\int\frac{d^D k}{(2\pi)^D}\frac{i}{k^2 - M^2}\frac{i}{(k-q)^2 - M^2}\mathrm{tr}\left[\slashed{k}(\slashed{k} - \slashed{q})\right]\right],$$

where Δ'^2 is short for $\Delta'(q)\Delta'(-q)$ and $|\Delta'|^2$ for $\Delta'^{*}(q)\Delta'(q)$. After some algebra this becomes

$$\int\frac{d^D q}{(2\pi)^D}\frac{1}{2}\left\{M^2(\Delta'^2 + \Delta'^{\dagger 2})\tilde{\Pi}(q_E^2/M^2) + 2|\Delta'|^2\left[\Pi(q_E^2/M^2) - M^2\tilde{\Pi}(q_E^2/M^2)\right]\right\},$$

where $\Pi(q_E^2/M^2)$ is the previous self-energy (23), and $\tilde{\Pi}(q_E^2/M^2)$ is the function

$$\tilde{\Pi}(q_E^2/M^2) = i2^{D/2}\int\frac{d^D k}{(2\pi)^D}\frac{i}{k^2 - M^2}\frac{i}{(k-q)^2 - M^2}$$

$$= \frac{D}{2}b_\epsilon(1 - D/2)\int_0^1 dx\left[q_E^2 x(1-x) + M^2\right]^{D/2 - 2}. \tag{43}$$

As a result, the action for the quadratic deviations from $\Delta_0 = M$ can be written as

$$\frac{1}{N}\delta^2\Gamma = -\frac{1}{2}\int\frac{d^D q}{(2\pi)^D}\left[\left(\frac{1}{g_0} + A\right)|\Delta'|^2 + \frac{1}{2}B(\Delta'^2 + \Delta'^{*2})\right], \tag{44}$$

with the coefficients

$$A = -\frac{D}{2}b_\epsilon M^\epsilon\left[(D-1)J_1^\epsilon(q_E^2/M^2) - (D/2 - 1)J_2^\epsilon(q_E^2/M^2)\right],$$

$$B = -\frac{D}{2}b_\epsilon M^\epsilon(D/2 - 1)J_2^\epsilon(q_E^2/M^2), \tag{45}$$

and the integrals

$$J_1^\epsilon(z) = \int_0^1 d^D x\left[zx(1-x) + 1\right]^{D/2 - 1}, \quad J_2^\epsilon(z) = \int_0^1 d^D x\left[zx(1-x) + 1\right]^{D/2 - 2}. \tag{46}$$

Thus the propagators of real and imaginary parts of the field $\Delta'(x)$ are in momentum space

$$G_{\Delta'_{\mathrm{re}}\Delta'_{\mathrm{re}}} = -\frac{i}{N}\frac{1}{1/g_0 + A + B}, \quad G_{\Delta'_{\mathrm{im}}\Delta'_{\mathrm{im}}} = -\frac{i}{N}\frac{1}{1/g_0 + A - B}. \tag{47}$$

The excitation spectrum is given by the zeros of the denominator functions

$$\frac{1}{g_0} + A + B = \frac{D}{2}b_\epsilon M^\epsilon\left[1 - (D-1)J_1^\epsilon(q_E^2/M^2)\right], \tag{48}$$

$$\frac{1}{g_0} + A - B = \frac{D}{2}b_\epsilon M^\epsilon\left\{\left[1 - (D-1)J_1^\epsilon(q_E^2/M^2)\right] + (D-2)J_2^\epsilon(q_E^2/M^2)\right\}. \tag{49}$$

By expanding $J_1^\epsilon(z), J_2^\epsilon(z)$ in powers of $z = q_E^2/M^2 \approx 0$,

$$J_1^\epsilon(z) \sim 1 + \frac{D-2}{12}z + \mathcal{O}(z^2), \quad J_2^\epsilon(z) \sim 1 + \frac{D-4}{12}z + \mathcal{O}(z^2), \tag{50}$$

we find

$$\frac{1}{g_0} + A + B = -\frac{D}{2} b_\epsilon M^\epsilon (D-2) \left(1 + \frac{D-1}{12} z\right) + \mathcal{O}(z^2),$$

$$\frac{1}{g_0} + A - B = -\frac{D}{2} b_\epsilon M^\epsilon \frac{D-2}{12} 3z + \mathcal{O}(z^2), \tag{51}$$

Inserting here the gap equation (19), we obtain

$$\frac{1}{g_0} + A + B = \epsilon \mu^\epsilon \left(\frac{1}{g^*} - \frac{1}{g}\right)\left(1 + \frac{D-1}{12} \frac{q_E^2}{M^2}\right) + \dots, \quad \frac{1}{g_0} + A - B = \epsilon \mu^\epsilon \left(\frac{1}{g^*} - \frac{1}{g}\right) \frac{1}{4} \frac{q_E^2}{M^2} + \dots. \tag{52}$$

Recalling (25) we see that the propagator of Δ'_{re} coincides with that of Σ' in the standard Gross-Neveu model, so that the fluctuations of Δ'_{re} have the same correlation length (26), with at a critical exponent ν in (27) as g approaches g^*.

In contrast, the propagator of the imaginary part of Δ' has now a pole at $q^2 = 0$:

$$G_{\Delta'_{\mathrm{im}} \Delta'_{\mathrm{im}}} = \frac{1}{N} \frac{4}{\epsilon} \left(\frac{1}{g^*} - \frac{1}{g}\right)^{-1} M^2 \frac{i}{q^2} + \text{regular part at } q^2 = 0. \tag{53}$$

The sign of the pole term guarantees a positive norm of the corresponding particle state in the Hilbert space. The particle is a Nambu-Goldstone boson.

4. Second phase transition

We are now prepared to show that the pair version of the chiral Gross-Neveu model in $2 + \epsilon$ dimensions has two phase transitions. Consider first the case $\epsilon = 0$ where the collective field theory consists of complex field Δ with O(2)-symmetry $\Delta \to e^{i\phi}\Delta$. From the work of Kosterlitz and Thouless [5] we know that such a field system possesses macroscopic excitations of the form of vortices and antivortices. These attract each other by a logarithmic Coulomb potential, just like a gas of electrons an positrons in two dimensions. At low temperatures, the vortices and antivortices form bound pairs. The grand-canonical ensemble of pairs exhibits quasi-long-range correlations. At some temperature T_c, the vortex pairs break up, and the correlations becomes short-range. The phase transition is of infinite order.

This transition is most easily understood in a model field theory involving of a pure phase field $\theta(x)$, with a Lagrange density

$$\mathscr{L} = \frac{\beta}{2} [\partial\theta(x)]^2, \tag{54}$$

where β is the stiffness of the θ-fluctuations. The important feature of the phase field θ is that it is a cyclic field with $\theta = \theta + 2\pi$. In order to ensure that such jumps by 2π carry no energy, the gradient in the Lagrange density needs a modification which allows for the existence of vortices and antivortices. This will not be discussed here in detail, since the reader may consult the literature for it [6,7]. We only state here that after including vortices and antivortices at positions x_i, x_j, their partition function can be written as

$$Z = \sum_{\mathrm{gas}} \exp\left\{4\pi^2\beta \sum_{i<j} q_i q_i \frac{1}{2\pi} \log\big(|x_i - x_j|/r_0\big)\right\}, \tag{55}$$

where r_0 is the size of the vortices. For a single vortex-antivortex pair, the average square distance r^2 is

$$\langle r^2 \rangle \propto \int_{r_0}^{\infty} dr\, r\, r^2 \exp\{-2\pi\beta \log(r/r_0)\} \propto \frac{1}{2\pi\beta - 4}. \tag{56}$$

This diverges as the stiffness falls below $\beta_{\mathrm{KT}} = 2/\pi \approx 0.63662$. A more detailed study shows that this is an exact result for a very dilute system of vortices and antivortices [7].

The large-stiffness state with bound vortex pairs has a coherent phase field $\theta(x)$, the low-stiffness state with separated vortex pairs exhibits incoherent phase fluctuations. The same situation is found in three dimensions, only that the excitations are vortex lines. These become infinitely long and prolific in a second-order phase transition [7] at a critical point $\beta_c \approx 0.33$.

The result (53) for $\epsilon = 0$ can now be used to estimate a critical value of the number of field components $N = N_c$ below which the phase fluctuations of the complex field Δ' become so violent that the system has a phase transition. For this we write $\Delta'_{\mathrm{im}} = M\theta$ and find from (53) a propagator of the θ-field

$$G_{\theta\theta} \approx \frac{i}{N} \frac{4\pi}{q^2} + \text{regular terms}. \tag{57}$$

Comparing this with the propagator for the model Lagrange density (54)

$$G_{\theta\theta} = \frac{1}{\beta} \frac{i}{q^2} \tag{58}$$

we identify the stiffness $\beta = N/4\pi$. The pair version of the chiral Gross-Neveu model has therefore a vortex-antivortex pair breaking transition if N falls below the critical value $N_c = 8$.

Consider now the model in $2 + \epsilon$ dimensions where pairs form at $g = g^* \approx \pi\epsilon$. A comparison between the propagator (53) and (58) yields a stiffness of phase fluctuations

$$\beta = \frac{N}{4\pi}\left(1 - \frac{g^*}{g}\right). \tag{59}$$

The linear vanishing of the stiffness with the distance of the coupling constant g from the critical value g^* is in agreement with a general scaling relation, according to which the critical exponent of bending rigidity should be equal to $(D-2)\nu$. Since the model has $\nu = 1/\epsilon$ [see (27)], this yields $(D-2)\nu = 1$, which is precisely the exponent in Eq. (59).

The stiffness (59) implies the existence of a phase transition in the neighborhood of two and in three dimensions at roughly

$$N_c \approx 8\left(1 - \frac{g^*}{g}\right)^{-1}, \quad D \approx 2,$$

$$N_c \approx 4.19\left(1 - \frac{g^*}{g}\right)^{-1}, \quad D = 3. \tag{60}$$

As N is lowered below these critical values, the phase fluctuations of the pair field Δ become incoherent and the pair condensate dissolves. The different phases are indicated in Fig. 1. In the chiral formulation of the same model, the intermediate phase has chiral symmetry in spite of a nonzero spontaneously generated ''quark mass'' $M \neq 0$. The reason why this is possible is that the ''quark mass'' depends only on $|\Delta_0|$, thus allowing for arbitrary phase fluctuations preserving chiral symmetry.

The sceptical reader may wonder whether the solid hyperbola in Fig. 1 is not simply the proper (albeit approximate) continuation of the vertical line for smaller N. There are two simple counterarguments. One is
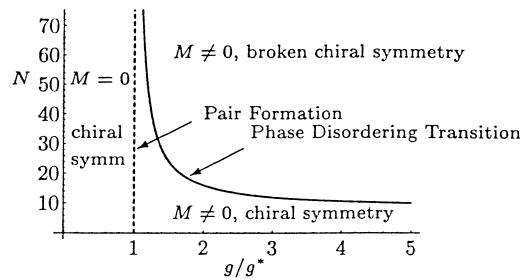
Fig. 1. The two transition lines in the $N - g$-plane of the chiral Gross-Neveu model in $2 + \epsilon$ dimensions. For $\epsilon = 0$, the vertical transition line coincides with the N-axis, and the solid hyperbola degenerates into a horizontal line at $N_c = 8$. The quark masses and chiral properties are indicated.

formal: For infinitesimal ϵ the first transition lies precisely at $g = g^* = \pi\epsilon$ for *all N*, so that the horizontal transition line is clearly distinguished from it. The other argument is physical. If N is lowered at some very large g, the binding energy of the pairs *increases with* $1/N$ {in two dimensions, the binding energy is $4M\sin^2[\pi/2(N-1)]$}. It is then impossible that the phase fluctuations on the horizontal branch of the transition line, which are low-energy excitations, unbind the strongly bound pairs. This will only happen in the limit $N \to \infty$ where the binding energy becomes zero and the two transition curves merge into a single curve. This is the situation in the BCS theory of superconductivity, where Cooper pair binding and pair condensation coincide.

In the ordinary Gross-Neveu model, the analog of the phase disordering transition is an Ising transition, in which the vacuum expectation value of σ jumps between $-\Sigma_0$ and Σ_0 in a disorderly fashion. In two dimensions, this occurs at some critical value N_c. In $2 + \epsilon$ dimensions, this transition should again exist independently of the transition at which the system enters into a state of nonzero Σ_0. It will be interesting to see these two transitions in either model confirmed by Monte-Carlo simulations.

Acknowledgements

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