Recursive graphical construction of Feynman diagrams in quantum electrodynamics

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We present a method for a recursive graphical construction of Feynman diagrams with their correct multiplicities in quantum electrodynamics. The method is first applied to find all diagrams contributing to the vacuum energy from which all *n*-point functions are derived by functional differentiation with respect to electron and photon propagators, and to the interaction. The basis for our construction is a functional differential equation obeyed by the vacuum energy when considered as a functional of the free propagators and the interaction. Our method does not employ external sources in contrast with traditional approaches.

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I. INTRODUCTION

In quantum field theory, it is well known [1,2] that the complete knowledge of all vacuum diagrams implies the knowledge of the entire theory (''the vacuum is the world''). Indeed, it is possible to derive all correlation functions and scattering amplitudes from the vacuum diagrams. This has been elaborated explicitly for ϕ^4 theory in the disordered phase in Refs. [3], [4] and for the ordered phase in Ref. [5], following a general theoretical framework laid out some time ago [6,7]. However, this knowledge has not yet been applied for constructing an efficient algebraic method along these lines for field theories of fundamental particles. The purpose of the present paper is to do this for quantum electrodynamics (QED). We show how to derive systematically all Feynman diagrams of the theory together with their correct multiplicities in a two step process: First we find the vacuum energy from a sum over all vacuum diagrams by a recursive graphical procedure. This is developed by solving a functional differential equation which involves functional derivatives with respect to the free electron and photon propagators. In a second step, we find all correlation functions by a diagrammatic application of functional derivatives upon the vacuum energy. In contrast with conventional procedures [8–13], no external currents coupled to single fields are used, such that there is no need for Grassmann sources for the electron fields. An additional advantage of our procedure is that the number of derivatives to be performed for a certain correlation function is half as big as with external sources.

In Sec. II we establish the partition function of Euclidean QED as a functional with respect to the inverse electron and photon progators as well as a generalized interaction. By setting up graphical representations for functional derivatives with respect to these bilocal and trilocal functions, we show in Sec. III that the partition function constitutes a generating functional for all correlation functions. This forms the basis for a perturbative expansion of the vacuum energy in terms of connected vacuum diagrams. In Sec. IV we then derive a recursion relation which allows to graphically construct the connected vacuum diagrams order by order. From these we obtain in Sec. V all diagrams for self-interactions and scattering processes by cutting electron as well as photon lines or by removing vertices. Along similar lines we apply in Sec. VI our method for scattering processes in the presence of an external electromagnetic field.

II. GENERATING FUNCTIONAL WITHOUT PARTICLE SOURCES

We begin by setting up a generating functional for all Feynman diagrams of quantum electrodynamics which does not employ external particle sources coupled linearly to the fields.

A. Partition function of QED

Our notation for the action of QED in Euclidean spacetime with a gauge fixing of Feynman type is

$$\mathcal{A}_{\rm QED}[\bar{\psi},\psi,A]=\int d^4x\Big[\bar{\psi}_\alpha(i\gamma^\mu_{\alpha\beta}\partial_\mu+m)\psi_\beta + \frac{1}{2}A_\mu(-\partial^2)A^\mu-e\bar{\psi}_\alpha\gamma^\mu_{\alpha\beta}A_\mu\psi_\beta\Big], \tag{2.1}$$

with Dirac spinor fields ψ_α, $\bar{\psi}_\beta=\psi^\dagger_\alpha\gamma^0_{\alpha\beta}$ $(\alpha,\beta=1,\ldots,4)$ and Maxwell's vector field A_μ $(\mu=0,\ldots,3)$. The properties of the vacuum are completely described by the partition function

$$Z_{\rm QED}=\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}Ae^{-\mathcal{A}_{\rm QED}[\bar{\psi},\psi,A]}, \tag{2.2}$$

where the electron fields $\bar{\psi}$ and ψ are Grassmannian. Let us split the action into the three terms

$$\mathcal{A}_{\rm QED}[\bar{\psi},\psi,A]=\mathcal{A}_\psi[\bar{\psi},\psi]+\mathcal{A}_A[A]+\mathcal{A}_{\rm int}[\bar{\psi},\psi,A], \tag{2.3}$$

corresponding to the Dirac, Maxwell, and interaction terms in Eq. (2.1). For the upcoming development it will be useful to consider the free parts of the action as bilocal functionals. The free action of the Dirac fields is

$$\mathcal{A}_\psi[\bar{\psi},\psi]=\int\int d^4x d^4x'\,\bar{\psi}_\alpha(x)S^{-1}_{F\alpha\beta}(x,x')\psi_\beta(x'), \tag{2.4}$$

with a kernel

$$S^{-1}_{F\alpha\beta}(x,x')=(i\gamma^\mu_{\alpha\beta}\partial_\mu+m\delta_{\alpha\beta})\delta(x-x'), \tag{2.5}$$

while the free action for the Maxwell field reads

$$\mathcal{A}_A[A]=\frac{1}{2}\int\int d^4x d^4x' A^\mu(x)D^{-1}_{F\mu\nu}(x,x')A^\nu(x') \tag{2.6}$$

with a kernel

$$D^{-1}_{F\mu\nu}(x,x')=-\partial^2\delta(x-x')\delta_{\mu\nu}. \tag{2.7}$$

In the following, we shall omit all vector and spinor indices, for brevity.

B. Generalized action

Our generating functional will arise from a generalization of the free action

$$\mathcal{A}^{(0)}[\bar{\psi},\psi,A]=\mathcal{A}_\psi[\bar{\psi},\psi]+\mathcal{A}_A[A] \tag{2.8}$$

to bilocal functionals of arbitrary kernels $S^{-1}(x_1,x_2)$ and $D^{-1}(x_1,x_2)=D^{-1}(x_2,x_1)$ according to

$$\mathcal{A}_\psi[\bar{\psi},\psi]\to\mathcal{A}_\psi[\bar{\psi},\psi;S^{-1}]=\int\int d^4x_1 d^4x_2\,\bar{\psi}(x_1)\times S^{-1}(x_1,x_2)\psi(x_2), \tag{2.9}$$

$$\mathcal{A}_A[A]\to\mathcal{A}_A[A;D^{-1}]=\frac{1}{2}\int\int d^4x_1 d^4x_2\times A(x_1)D^{-1}(x_1,x_2)A(x_2). \tag{2.10}$$

The kernels $S^{-1}(x_1,x_2)$ and $D^{-1}(x_1,x_2)$ are only required to possess a functional inverse $S(x_1,x_2)$ and $D(x_1,x_2)$. Similarly, we shall generalize the interaction to

$$\mathcal{A}_{\rm int}[\bar{\psi},\psi,A]\to\mathcal{A}_{\rm int}[\bar{\psi},\psi,A;V]=-e\int\int\int d^4x_1 d^4x_2 d^4x_3 V(x_1,x_2;x_3)\times\bar{\psi}(x_1)\psi(x_2)A(x_3), \tag{2.11}$$

where $V(x_1,x_2;x_3)$ is an arbitrary trilocal function. At the end we shall return to QED by substituting $S\to S_F$, $D\to D_F$ and $eV(x_1,x_2;x_3)\to e\gamma_\mu\delta(x_1-x_2)\delta(x_1-x_3)$.

The generalized partition function

$$Z=\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]} \tag{2.12}$$

with the action

$$\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]=\mathcal{A}_\psi[\bar{\psi},\psi;S^{-1}]+\mathcal{A}_A[\bar{\psi},\psi;D^{-1}]+\mathcal{A}_{\rm int}[\bar{\psi},\psi,A;V] \tag{2.13}$$

then represents a functional of the bilocal quantities $S^{-1}(x_1,x_2)$, $D^{-1}(x_1,x_2)$, and of the trilocal function $V(x_1,x_2;x_3)$. All n-point correlation functions of the theory are obtained from expectation values defined by

$$\langle\hat{O}_1(x_1)\hat{O}_2(x_2)\cdots\rangle=Z^{-1}\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A O_1(x_1)\times O_2(x_2)\cdots e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}, \tag{2.14}$$

where the local operators $\hat{O}_i(x)$ are products of field operators $\hat{\psi}(x)$, $\hat{\bar{\psi}}(x)$, and $\hat{A}(x)$ at the same spacetime point. Important examples for expectation values of this kind are the photon and the electron propagators of the interacting theory

$$^\gamma G^2(x_1,x_2)\equiv\langle\hat{A}(x_1)\hat{A}(x_2)\rangle=Z^{-1}\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\;A(x_1)A(x_2)\times e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}, \tag{2.15}$$

$$^e G^2(x_1,x_2)\equiv\langle\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\rangle=Z^{-1}\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\,\psi(x_1)\bar{\psi}(x_2)\times e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}. \tag{2.16}$$

For a perturbative calculation of the partition function Z we define the free vacuum functional

$$Z^{(0)}\equiv\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A e^{-\mathcal{A}^{(0)}[\bar{\psi},\psi,A;S^{-1},D^{-1}]}, \tag{2.17}$$

whose action is quadratic in the fields. The path integral is Gaussian and yields

$$Z^{(0)}=\exp[\mathrm{Tr}\ln S^{-1}]\exp\left[-\frac{1}{2}\mathrm{Tr}\ln D^{-1}\right]. \tag{2.18}$$

The free correlation functions of arbitrary local electron and photon operators $\hat{O}(x)$ are defined by the free part of the expectation values (2.14)

$$\langle\hat{O}_1(x_1)\hat{O}_2(x_2)\cdots\rangle^{(0)}=[Z^{(0)}]^{-1}\oint\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A O_1(x_1)\times O_2(x_2)\cdots e^{-\mathcal{A}^{(0)}[\bar{\psi},\psi,A;S^{-1},D^{-1}]}, \tag{2.19}$$

and the free-field propagators are the expectation values

$$^\gamma G^{2^{(0)}}(x_1,x_2)=D(x_1,x_2)\equiv\langle\hat{A}(x_1)\hat{A}(x_2)\rangle^{(0)}\equiv D(x_2,x_1), \tag{2.20}$$

$$ {}^{e}G^{2^{(0)}}(x_1,x_2)=S(x_1,x_2)\equiv\langle\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)\rangle^{(0)}. \tag{2.21}$$

To avoid a pile up of infinite volume factors in a perturbation expansion, it is favorable to go over from $Z^{(0)}$ to the negative vacuum energy $W^{(0)}$ defined by

$$W^{(0)}\equiv\ln Z^{(0)}=W_{\psi}^{(0)}+W_{A}^{(0)}, \tag{2.22}$$

where the free electron and photon parts are

$$W_{\psi}^{(0)}=\mathrm{Tr}\ln S^{-1} \tag{2.23}$$

and

$$W_{A}^{(0)}=-\frac{1}{2}\mathrm{Tr}\ln D^{-1}. \tag{2.24}$$

The negative total vacuum energy

$$W=\ln Z \tag{2.25}$$

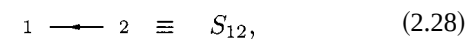
is obtained perturbatively by expanding the functional integral (2.12) in powers of the coupling constant e:

$$W=\sum_{p=0}^{\infty}e^{2p}W^{(p)}, \tag{2.26}$$

where the quantities $W^{(p)}$ with $p\geqslant 1$ are free-field expectation values of the type (2.19):

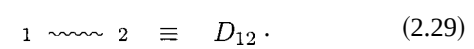
$$W^{(p)}=\int_{1\cdots 6p}V_{123}\cdots V_{6p-2\,6p-1\,6p}\times\langle\overline{\hat{\psi}}_{6p-1}\cdots\hat{\psi}_5\hat{\psi}_2\hat{\bar{\psi}}_1\hat{\bar{\psi}}_4\cdots\hat{\bar{\psi}}_{6p-2}\hat{A}_3\hat{A}_6\cdots\hat{A}_{6p}\rangle^{(0)}, \quad p\geqslant 1. \tag{2.27}$$

Throughout this paper we shall use from now on the shorthand notation $1=x_1$, $2=x_2,\ldots$ and $\int_{1\cdots}=\int d^4x_1\cdots$. The expectation values in Eq. (2.27) are evaluated with the help of Wick's rule as a sum of Feynman integrals, which are pictured as connected vacuum diagrams constructed from lines and vertices. A straight line with an arrow represents an electron propagator



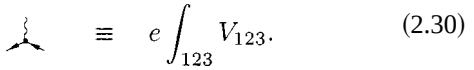
$$\equiv\quad S_{12}, \tag{2.28}$$

whereas a wiggly line stands for a photon propagator



$$\equiv\quad D_{12}\,. \tag{2.29}$$

The vertex represents an integral over the interaction potential:



$$\equiv\quad e\int_{123}V_{123}. \tag{2.30}$$

The vacuum energies (2.23) and (2.24) will be represented by single-loop diagrams

$$W_{\psi}^{(0)}=- \tag{2.31}$$

and

$$W_{A}^{(0)}=\frac{1}{2} \tag{2.32}$$

This leaves us with the important problem of finding all connected vacuum diagrams. For this we shall exploit that the partition function (2.12) is a functional of the bilocal functions $S^{-1}(x_1,x_2)$, $D^{-1}(x_1,x_2)$, and of the trilocal function $V(x_1,x_2;x_3)$.

III. PERTURBATION THEORY

As a preparation for our generation procedure for vacuum diagrams, we set up a graphical representation of functional derivatives with respect to the kernels S^{-1}, D^{-1}, the propagators S, D, and the interaction function V. After this we express the vacuum functional W in terms of a series of functional derivatives of the free partition function $Z^{(0)}$ with respect to the kernels.

A. Functional derivatives with respect to $S^{-1}(x_1,x_2)$, $D^{-1}(x_1,x_2)$, and $V(x_1,x_2;x_3)$

Each Feynman diagram is composed of integrals over products of the propagators S,D and may thus be considered as a functional of the kernels S^{-1},D^{-1}. In the following we set up the graphical rules for performing functional derivatives with respect to these functional matrices. With these rules we can generate all $2n$-point correlation functions with $n=1,2,\ldots$ from vacuum diagrams. To produce also $(2n+1)$-point correlation functions with $n=1,2,\ldots$ such as the fundamental three-point vertex function from vacuum diagrams, it is useful to introduce additionally a functional derivative with respect to the interaction function V.

1. Functional derivative with respect to the photon kernel

The kernel D_{12}^{-1} of the photon is symmetric $D_{12}^{-1}=D_{21}^{-1}$, so that the basic functional derivatives are also symmetric [3,4]

$$\frac{\delta D_{12}^{-1}}{\delta D_{34}^{-1}}\equiv\frac{1}{2}\{\delta_{13}\delta_{42}+\delta_{14}\delta_{32}\}, \tag{3.1}$$

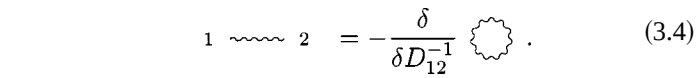
as is discussed in detail in Ref. [5]. By the chain rule of differentiation, this defines the functional derivative with respect to D^{-1} for all functionals of D^{-1}. As an example, we calculate the free photon propagator (2.20) by applying the operator $\delta/\delta D_{12}^{-1}$ to Eq. (2.17). Taking into account Eq. (2.22) and Eq. (3.1), we find

$$D_{12}=-2\frac{\delta W_{A}^{(0)}}{\delta D_{12}^{-1}}. \tag{3.2}$$

Inserting the explicit form (2.24), we obtain

$$D_{12}=\frac{\delta}{\delta D_{12}^{-1}}\,\mathrm{Tr}\ln D^{-1}. \tag{3.3}$$

With the notation (2.29) and (2.32), we can write relation (3.2) graphically as



(3.4)

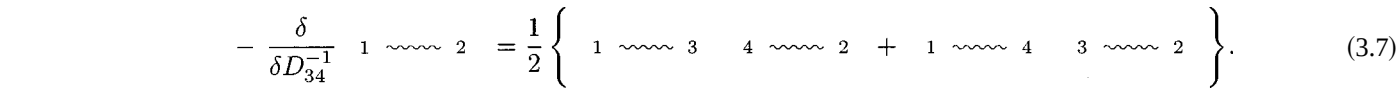
This diagrammatic equation may be viewed as a special case of a general graphical rule derived as follows: Let us apply the functional derivative (3.1) to a photon propagator D_{12}. Because of the identity

$$\int_1 D_{11}\frac{\delta D_{12}^{-1}}{\delta D_{34}^{-1}}=\delta_{12} \tag{3.5}$$

we find

$$-\frac{\delta D_{12}}{\delta D_{34}^{-1}}=\frac{1}{2}\{D_{13}D_{42}+D_{14}D_{32}\}. \tag{3.6}$$

Diagrammatically, this equation implies that the operation $-\delta/\delta D_{34}^{-1}$ applied to a photon line (2.29) amounts to cutting the line:



(3.7)

Note that the indices of the kernel D_{34}^{-1} are symmetrically attached to the newly created line ends in the two possible ways due to the differentiation rule (3.1). This rule implies directly the diagrammatic equation (3.4).

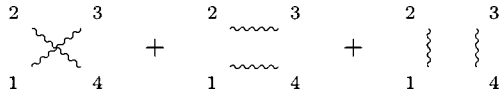
Consider now higher-order correlation functions which follow from higher functional derivatives of $W_A^{(0)}$. From the definition (2.19) and Eq. (2.10), we obtain the free four-point function as the second functional derivative

$${}^{\gamma}G_{1234}^{4(0)}\equiv\langle\hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4\rangle^{(0)}=4e^{-W_A^{(0)}}\frac{\delta^2}{\delta D_{12}^{-1}\delta D_{34}^{-1}}e^{W_A^{(0)}}. \tag{3.8}$$

Because of the symmetry of D_{12}, the order in which the spacetime arguments appear in the inverse propagators is of no importance. Inserting for $W_A^{(0)}$ the explicit form (2.24), the first derivative yields via Eq. (3.3) just $-D_{34}\exp\{W_A^{(0)}\}$, the second derivative applied to this gives with the rule (3.6) and, once more Eq. (3.3),

$${}^{\gamma}G_{1234}^{4(0)}=D_{13}D_{24}+D_{32}D_{14}+D_{12}D_{34}. \tag{3.9}$$

The right-hand side has the graphical representation



(3.10)

The same diagrams are obtained by applying the cutting rule (3.7) twice to the single-loop diagram (2.32).

While derivatives with respect to the kernel D^{-1} amount to cutting photon lines, we show now that derivatives with respect to the photon propagator D amount to line amputations. The transformation rule between the two operations follows from relation (3.6):

$$\frac{\delta}{\delta D_{12}^{-1}}=-\int_{34}D_{13}D_{24}\frac{\delta}{\delta D_{34}}, \tag{3.11}$$

which is equivalent to

$$\frac{\delta}{\delta D_{12}}=-\int_{34}D_{13}^{-1}D_{24}^{-1}\frac{\delta}{\delta D_{34}^{-1}}. \tag{3.12}$$

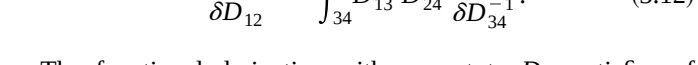
The functional derivative with respect to D_{12} satisfies of course the fundamental relation (3.1):

$$\frac{\delta D_{12}}{\delta D_{34}}=\frac{1}{2}\{\delta_{13}\delta_{42}+\delta_{14}\delta_{32}\}. \tag{3.13}$$

We shall represent the right-hand side graphically by extending the Feynman diagrams by the symbol:

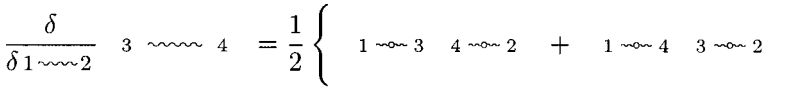
$\equiv\quad\delta_{12}.$ (3.14)

If we write the functional derivative with respect to the propagator D_{12} graphically as



(3.15)

we may express Eq. (3.13) as



(3.16)

Thus, differentiating a photon line with respect to the corresponding propagator amputates this line, leaving only the symmetrized indices at the end points.

2. Functional derivative with respect to the electron kernel

Setting up graphical representations for functional derivatives for electrons is different from that in the photon case since the kernel S^{-1} is no longer symmetric. The functional derivative is therefore the usual one

$$\frac{\delta S_{12}^{-1}}{\delta S_{34}^{-1}} = \delta_{13}\delta_{42}, \tag{3.17}$$

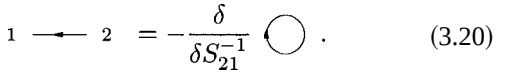
from which all others are derived via the chain rule of differentiation. The free electron propagator S_{12} is found in analogy to Eq. (3.2) by differentiating the free electron vacuum functional (2.23) with respect to the inverse electron propagator S^{-1}:

$$S_{12} = \frac{\delta W_{\psi}^{(0)}}{\delta S_{21}^{-1}}. \tag{3.18}$$

This implies

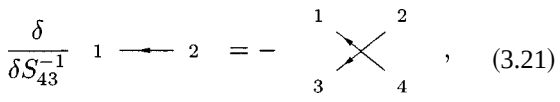
$$S_{12} = \frac{\delta}{\delta S_{21}^{-1}} \mathrm{Tr} \ln S^{-1}, \tag{3.19}$$

which follows also from Eq. (3.17) and the chain rule of differentiation. The graphical interpretation of the functional derivative $\delta / \delta S_{21}^{-1}$ is quite analogous to the photon case. In analogy to Eq. (3.4), we write expression (3.19) diagrammatically as



(3.20)

This, in turn, can be understood as being a consequence of the general cutting rule for electron lines:



(3.21)

which graphically expresses the derivative relation

$$\frac{\delta S_{12}}{\delta S_{43}^{-1}} = -S_{14}S_{32}. \tag{3.22}$$

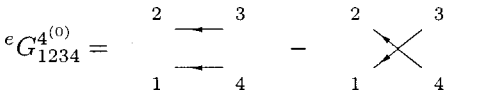
The free electron 4-point function is obtained from two functional derivatives according to

$${}^{e}G_{1234}^{4(0)} \equiv \langle \hat{\psi}_1 \hat{\psi}_2 \bar{\hat{\psi}}_3 \bar{\hat{\psi}}_4 \rangle^{(0)} = e^{-W_{\psi}^{(0)}} \frac{\delta^2}{\delta S_{32}^{-1} \delta S_{41}^{-1}} e^{W_{\psi}^{(0)}}. \tag{3.23}$$

Here, the electron fields must be properly rearranged to $\hat{\psi}_2 \bar{\hat{\psi}}_3 \hat{\psi}_1 \bar{\hat{\psi}}_4$ for applying the functional derivatives with respect to S^{-1}. Using Eqs. (3.18) and (3.22) we obtain, from Eq. (3.23),

$${}^{e}G_{1234}^{4(0)} = S_{23}S_{14} - S_{24}S_{13}, \tag{3.24}$$

or graphically

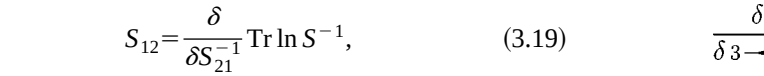


(3.25)

Derivatives with respect to the propagators S satisfy the relation

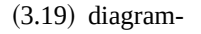
$$\frac{\delta S_{12}}{\delta S_{34}} = \delta_{13}\delta_{42}, \tag{3.26}$$

which in analogy to Eq. (3.13) is represented graphically as an amputation of an electron line



(3.27)

Here we have introduced the additional diagrammatic symbols



(3.28)

(3.29)

Differentiating an electron line with respect to the corresponding propagator removes this line, leaving only the indices at the end points of the remaining lines.

The analytic relations between cutting and amputating lines are now, just as in Eqs. (3.11) and (3.12):

$$\frac{\delta}{\delta S_{12}^{-1}} = -\int_{34} S_{31}S_{24} \frac{\delta}{\delta S_{34}}, \tag{3.30}$$

$$\frac{\delta}{\delta S_{12}} = -\int_{34} S_{31}^{-1}S_{24}^{-1} \frac{\delta}{\delta S_{34}^{-1}}. \tag{3.31}$$

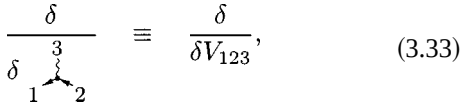
With the above graphical representations of the functional derivatives, it will be possible to derive systematically all vacuum diagrams of the interacting theory order by order in the coupling strength e, and from these all diagrams with an even number of legs.

3. Functional derivative with respect to the interaction

If we want to find amplitudes involving an odd number of photons such as the three-point function from vacuum diagrams, the derivatives with respect to the kernels S^{-1}, D^{-1} are not enough. Here the general trilocal interaction function V of Eq. (2.11) is needed. Thus, we define an associated functional derivative with respect to this interaction to satisfy

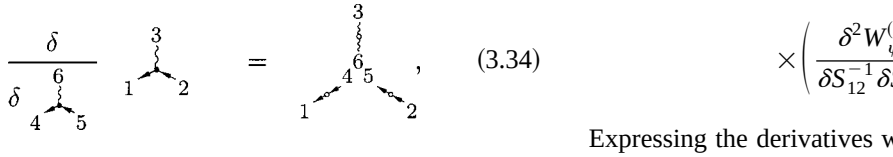
$$\frac{\delta V_{123}}{\delta V_{456}} = \delta_{14}\delta_{52}\delta_{36}. \tag{3.32}$$

By introducing the graphical rule



$$\equiv \quad \frac{\delta}{\delta V_{123}}, \tag{3.33}$$

the definition of the functional derivative (3.32) can be expressed as



$$= \tag{3.34}$$

where the right-hand side represents a product of δ-functions as defined in Eqs. (3.14) and (3.28).

B. Vacuum energy as generating functional

With the above-introduced diagrammatic operations, the vacuum energy $W[S^{-1},D^{-1},V]$ constitutes a generating functional for all correlation functions. Its evaluation proceeds by expanding the exponential in the partition function (2.12) in powers of the coupling constant e, leading to the Taylor series

$$\begin{aligned} Z = \sum_{p=0}^{\infty} \frac{e^{2p}}{(2p)!} \oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \\ \times \left(\int_{1\cdots 6} V_{123}V_{456}\bar{\psi}_1\psi_2 A_3 \bar{\psi}_4\psi_5 A_6 \right)^p \\ \times e^{-\mathcal{A}^{(0)}[\bar{\psi},\psi;A;S^{-1},D^{-1}]}. \end{aligned} \tag{3.35}$$

The products of pairs of fields $\bar{\psi}_1\psi_2$ and A_3A_6 can be substituted by a functional derivative with respect to S^{-1} and D^{-1}, leading to the perturbation expansion

$$\begin{aligned} Z &\equiv \sum_{p=0}^{\infty} e^{2p} Z^{(p)} \\ &= \sum_{p=0}^{\infty} \frac{(-2e^2)^p}{(2p)!} \left(\int_{1\cdots 6} V_{123}V_{456} \frac{\delta^3}{\delta S_{12}^{-1}\delta S_{45}^{-1}\delta D_{36}^{-1}} \right)^p Z^{(0)}. \end{aligned} \tag{3.36}$$

Note the two advantages of this expansion over the conventional one in terms of currents coupled linearly to the fields. First, it contains only half as many functional derivatives. Second, it does not contain derivatives with respect to Grassmann variables.

Inserting for $Z^{(0)}$ the free vacuum functional (2.22), we obtain for the first-order term $Z^{(1)}$

$$Z^{(1)} = \frac{1}{2!} \int_{1\cdots 6} V_{123}V_{456}(-2) \frac{\delta^3}{\delta D_{36}^{-1}\delta S_{12}^{-1}\delta S_{45}^{-1}} Z^{(0)}. \tag{3.37}$$

Since

$$Z = Z^{(0)} + e^2 Z^{(1)} + \cdots = \exp\{W^{(0)} + e^2 W^{(1)} + \cdots\}, \tag{3.38}$$

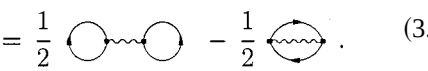
this corresponds to a first-order correction $W^{(1)}$ to the vacuum energy $W^{(0)}$:

$$\begin{aligned} W^{(1)} = \frac{1}{2!} \int_{1\cdots 6} V_{123}V_{456}(-2) \frac{\delta W_A^{(0)}}{\delta D_{36}^{-1}} \\ \times \left(\frac{\delta^2 W_\psi^{(0)}}{\delta S_{12}^{-1}\delta S_{45}^{-1}} + \frac{\delta W_\psi^{(0)}}{\delta S_{12}^{-1}} \frac{\delta W_\psi^{(0)}}{\delta S_{45}^{-1}} \right). \end{aligned} \tag{3.39}$$

Expressing the derivatives with respect to the kernels by the corresponding propagators via Eqs. (3.2), (3.18), and taking into account Eq. (3.22), $W^{(1)}$ becomes

$$W^{(1)} = \frac{1}{2} \int_{1\cdots 6} V_{123}V_{456}D_{36}(S_{21}S_{54} - S_{24}S_{51}). \tag{3.40}$$

According to the Feynman rules (2.28)–(2.30), this is represented by the diagrams



$$W^{(1)} = \frac{1}{2} \quad - \frac{1}{2} \tag{3.41}$$

Note that each closed electron loop causes a factor -1.

IV. GRAPHICAL RECURSION RELATION FOR CONNECTED VACUUM DIAGRAMS

In this section, we derive a functional differential equation for the vacuum functional $W[S^{-1},D^{-1},V]$ whose solution leads to a graphical recursion relation for all connected vacuum diagrams.

A. Functional differential equation for $W=\ln Z$

The functional differential equation for the vacuum functional $W[S^{-1},D^{-1},V]$ is derived from the following functional integral identity:

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A \frac{\delta}{\delta\bar{\psi}_1} \{\bar{\psi}_2 e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}\} = 0 \tag{4.1}$$

with the action (2.13). This identity is the functional generalization of the trivial integral identity $\int_{-\infty}^{+\infty} dx\, f'(x) = 0$ for functions $f(x)$ which vanish at infinity. Nontrivial conse-

quences of Eq. (4.1) are obtained by performing the functional derivative in the integrand which yields

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\left\{\delta_{12}+\int_3 \bar{\psi}_2 S^{-1}_{13}\psi_3 - e\int_{34} V_{134}\bar{\psi}_2\psi_3 A_4\right\} \times e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}=0. \tag{4.2}$$

Substituting the field products $\bar{\psi}_2\psi_3$ by functional derivatives with respect to the electron kernel S^{-1}_{23}, this equation can be expressed in terms of the partition function (2.12):

$$Z\delta_{12}-\int_3 S^{-1}_{13}\frac{\delta Z}{\delta S^{-1}_{23}}+e\int_{34} V_{134}\frac{\delta}{\delta S^{-1}_{23}}[\langle \hat{A}_4\rangle Z]=0. \tag{4.3}$$

To bring this functional differential equation into a more convenient form, we calculate explicitly the term containing the expectation of the field A. This is done starting from the integral identity

$$\oint \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A\,\frac{\delta}{\delta A_1}e^{-\mathcal{A}[\bar{\psi},\psi,A;S^{-1},D^{-1},V]}=0. \tag{4.4}$$

Note this identity is not endangered by the gauge freedom in the electromagnetic vector potential A_μ due to the presence of a gauge fixing term in the action (2.1). This ensures that the exponential vanishes at the boundary of all A field directions [14].

After differentiating the action in the exponential of Eq. (4.4), we find the expectation of the photon field

$$\int_1 \langle \hat{A}_1\rangle D^{-1}_{12}=-e\int_{34} V_{342}\langle \psi_4\bar{\psi}_3\rangle. \tag{4.5}$$

Multiplying this with $\int_2 D_{25}$, we yield

$$\langle \hat{A}_5\rangle=-e\int_{234} V_{342}D_{25}\frac{\delta W}{\delta S^{-1}_{34}}, \tag{4.6}$$

where we have used $Z=e^W$. Inserting this into Eq. (4.3), we obtain

$$\delta_{12}-\int_3 S^{-1}_{13}\frac{\delta W}{\delta S^{-1}_{23}}=e^2\int_{3\cdots 7} V_{134}V_{567}D_{47} \times\left\{\frac{\delta^2 W}{\delta S^{-1}_{23}\delta S^{-1}_{56}}+\frac{\delta W}{\delta S^{-1}_{23}}\frac{\delta W}{\delta S^{-1}_{56}}\right\}. \tag{4.7}$$

Setting $x_1=x_2$ and performing the integration over x_1, this leads to the nonlinear functional differential equation for the vacuum functional W

$$\int_1 \delta_{11}-\int_{12} S^{-1}_{12}\frac{\delta W}{\delta S^{-1}_{12}}=e^2\int_{1\cdots 6} V_{123}V_{456}D_{36} \times\left\{\frac{\delta^2 W}{\delta S^{-1}_{12}\delta S^{-1}_{45}}+\frac{\delta W}{\delta S^{-1}_{12}}\frac{\delta W}{\delta S^{-1}_{45}}\right\}, \tag{4.8}$$

which will form the basis for deriving the desired recursion relation for the vacuum diagrams. The first term on the left-hand side of Eq. (4.8) is infinite, but in the next section we will show that this cancels against an infinity in the second term.

B. Recursion relation

Equation (4.8) contains functional derivatives with respect to the electron kernel S^{-1} which are equivalent to cutting lines in the vacuum diagrams. For practical purposes it will be more convenient to work with derivatives with respect to the propagators S which remove electron lines. The second term on the left-hand side of Eq. (4.8) contains the operation $-\int_{12} S^{-1}_{12}\delta/\delta S^{-1}_{12}$, which we convert into the differential operator

$$\hat{N}_F=\int_{12} S_{12}\frac{\delta}{\delta S_{12}} \tag{4.9}$$

with the help of Eq. (3.30). This operator has a simple graphical interpretation. The derivative $\delta/\delta S_{12}$ removes an electron line from a Feynman diagram, and the factor S_{12} restores it. This operation is familiar from the number operator in second quantization. The operator $\hat{N}_F$ counts the number of electron lines in a Feynman diagram G:

$$\hat{N}_F G=N_F G. \tag{4.10}$$

When applied to the vacuum diagrams $W^{(p)}$ of order $p\geqslant 1$, this operator gives

$$\hat{N}_F W^{(p)}=2pW^{(p)},\quad p\geqslant 1, \tag{4.11}$$

since the number of electron lines in a vacuum diagram without external sources in quantum electrodynamics is equal to the number of vertices. The restriction in Eq. (4.11) to $p\geqslant 1$ is necessary due to a special role of the vacuum diagram. Take, for example, the electron vacuum diagram of the free theory (2.23). By applying the operator $\hat{N}_F$, we obtain with Eq. (3.18)

$$\hat{N}_F W^{(0)}_\psi=-\int_{12} S^{-1}_{12}S_{21}=-\int_1 \delta_{11}, \tag{4.12}$$

which is a divergent trace integral precisely canceling the infinite first term in Eq. (4.8).

Separating out $W^{(0)}$ in the expansion (2.26) of the vacuum functional, the left-hand side of the functional differential equation (4.8) has the expansion

$$\int_1 \delta_{11} + \hat{N}_F W = \sum_{p=1}^{\infty} 2pe^{2p} W^{(p)} = \sum_{p=0}^{\infty} 2(p+1)e^{2(p+1)} W^{(p+1)}. \tag{4.13}$$

On the right hand side of Eq. (4.8), we express the first and second functional derivatives with respect to the kernel S^{-1} in terms of functional derivatives with respect to the propagator S by using Eq. (3.30) and

$$\frac{\delta^2}{\delta S_{12}^{-1} \delta S_{34}^{-1}} = \int_{5678} S_{51} S_{26} S_{73} S_{48} \frac{\delta^2}{\delta S_{56} \delta S_{78}} + \int_{56} [S_{53} S_{41} S_{26} + S_{23} S_{46} S_{51}] \frac{\delta}{\delta S_{56}}. \tag{4.14}$$

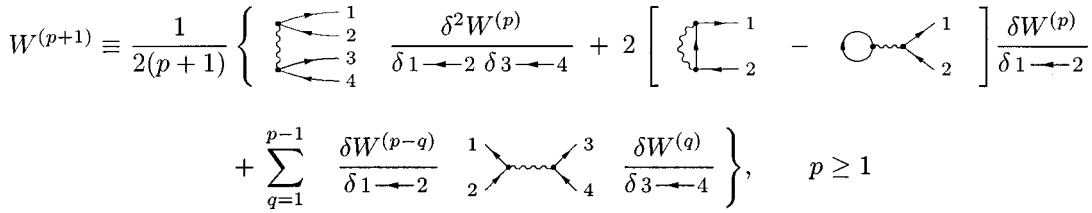
Inserting here the expansion (2.26) and comparing equal powers in e with those in Eq. (4.13), we obtain the following recursion formula for the expansion coefficients of the vacuum functional

$$W^{(p+1)} = \frac{1}{2(p+1)} \left\{ \int_{1\ldots10} V_{123} V_{456} D_{36} S_{71} S_{28} S_{94} S_{510} \frac{\delta^2 W^{(p)}}{\delta S_{78} \delta S_{910}} + 2 \int_{1\ldots8} V_{123} V_{456} D_{36} (S_{51} S_{28} S_{74} - S_{71} S_{28} S_{54}) \frac{\delta W^{(p)}}{\delta S_{78}} \right.$$
$$\left. + \sum_{q=1}^{p-1} \int_{1\ldots10} V_{123} V_{456} D_{36} S_{71} S_{28} S_{94} S_{510} \frac{\delta W^{(q)}}{\delta S_{78}} \frac{\delta W^{(p-q)}}{\delta S_{910}} \right\}, \quad p \geqslant 1 \tag{4.15}$$

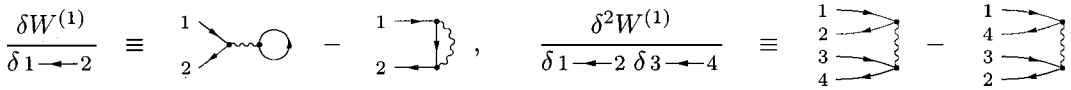
and the initial value (3.40). This equation enables us to derive the connected vacuum diagrams systematically to any desired order from the diagrams of the previous orders, as will now be shown.

C. Graphical solution

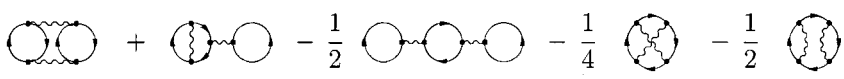
With the help of the Feynman rules (2.28)–(2.30), the functional recursion relation (4.15) can be written diagrammatically as follows:

$W^{(p+1)} \equiv$  $, \quad p \geq 1$ (4.16)

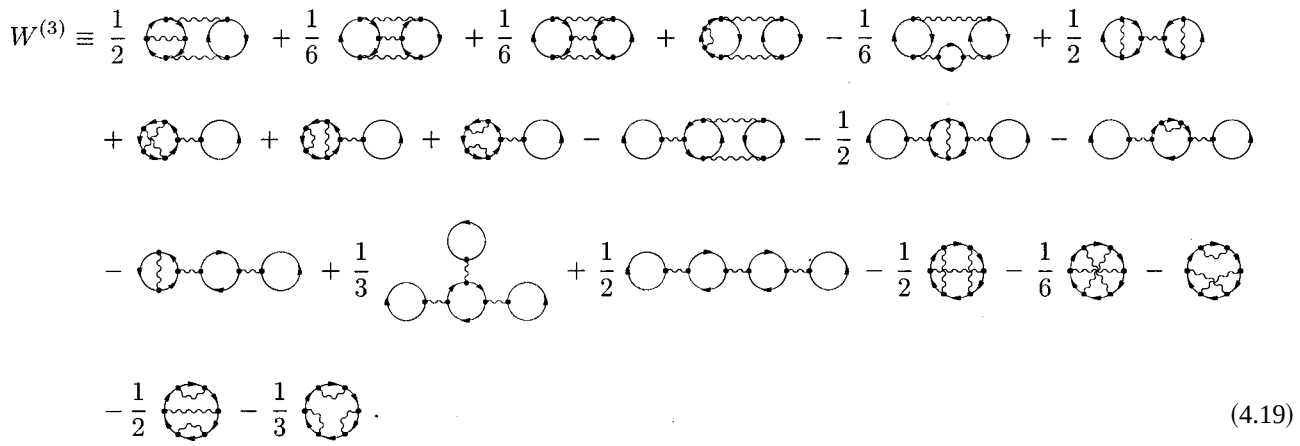
and the first-order result is given by Eq. (3.41). The right-hand side contains four graphical operations. The first three are linear and involve one or two electron line amputations of the previous perturbative order. The fourth operation is nonlinear and mixes two different electron line amputations of lower orders. To demonstrate the working of this formula, we calculate the connected vacuum diagrams in second and third order. We start with the amputation of one or two electron lines in first order (3.41):

 (4.17)

Inserting Eq. (4.17) into Eq. (4.16), where we have to take care of connecting only legs with the same label, we find the second-order correction of the vacuum functional W:

$W^{(2)} \equiv$  (4.18)

The calculation of the third-order correction $W^{(3)}$ leads to the following 20 diagrams:



(4.19)

From the vacuum diagrams (3.41), (4.18), and (4.19), we observe a simple mnemonic rule for the weights of the connected vacuum diagrams in QED. At least up to four loops, each weight is equal to the reciprocal number of electron lines, which, by cutting, generate the same two-point diagrams. The sign is given by $(-1)^L$, where L denotes the number of electron loops. Note that the total weight, which is the sum over all weights of the vacuum diagrams in the order to be considered, vanishes in QED. The simplicity of the weights is a consequence of the Fermi statistics and the three-point form of interaction (2.11). The weights of the vacuum diagrams in other theories, like ϕ^4-theory [3,4,6], follow more complicated rules.

V. SCATTERING BETWEEN ELECTRONS AND PHOTONS

From the above vacuum diagrams, we obtain all even-point correlation functions by cutting electron or photon lines. For the generation of the odd-point functions we use the functional derivative (3.34) with respect to the interaction function V which removes a vertex from a diagram.

As an illustration, we generate the diagrams for the self interactions described by the propagators (2.15) and (2.16)

$$ {}^{\gamma}G^2_{12}=\langle \hat{A}_1\hat{A}_2\rangle, \quad {}^{e}G^2_{12}=\langle \hat{\psi}_1\hat{\bar{\psi}}_2\rangle \tag{5.1}$$

and the four-point functions

$$ {}^{\gamma\gamma}G^4_{1234}=\langle \hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4\rangle, \quad {}^{ee}G^4_{1234}=\langle \hat{\psi}_1\hat{\psi}_2\hat{\bar{\psi}}_3\hat{\bar{\psi}}_4\rangle,$$

$$ {}^{e\gamma}G^5_{1234}=\langle \hat{\psi}_1\hat{A}_2\hat{A}_3\hat{\bar{\psi}}_4\rangle, \tag{5.2}$$

which represent the simplest scattering processes of the theory. In addition, we give the perturbative expansion of the three-point vertex function

$$ G^3_{123}=\langle \hat{\psi}_1\hat{\bar{\psi}}_2\hat{A}_3\rangle. \tag{5.3}$$

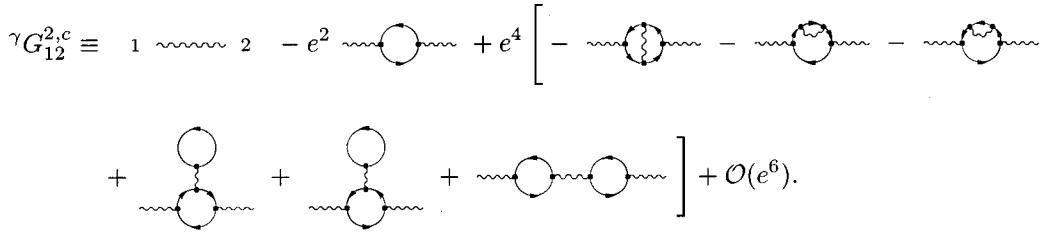
The following examples illustrate the simple weights $(-1)^L$ of diagrams contributing to an n-point function with $n\geqslant 2$, with L being the number of electron loops.

A. Self-interactions

Substituting the product of the photon fields A_1A_2 in the functional integral (2.15) by the photonic functional derivative $-2\,\delta/\delta D^{-1}_{12}$, the photonic two-point function of the interacting theory is given by

$$ {}^{\gamma}G^2_{12}=-2\,\frac{\delta}{\delta D^{-1}_{12}}\,W[S^{-1},D^{-1},V]. \tag{5.4}$$

Applying the associated cutting rule (3.7) to the vacuum diagrams (3.41) and (4.18) leads to the connected diagrams



(5.5)

For brevity, we have omitted the labels 1 and 2 at the ends of the higher-order diagrams. The full and the connected propagators ${}^{\gamma}G^{2}_{12}$ and ${}^{\gamma}G^{2,c}_{12}$ satisfy the cumulant relation

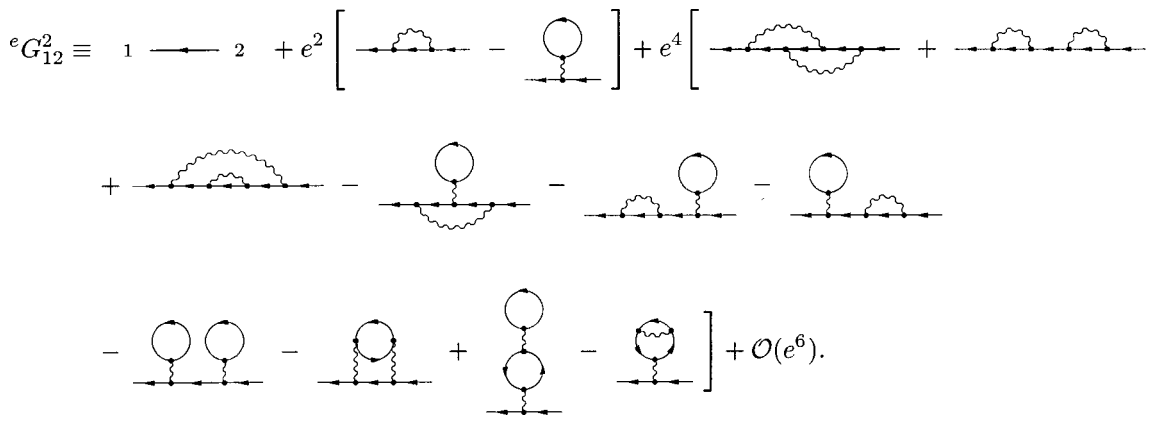
$$ {}^{\gamma}G^{2,c}_{12} = {}^{\gamma}G^{2}_{12} - \langle \hat{A}_1 \rangle \langle \hat{A}_2 \rangle . \tag{5.6} $$

Note that although the expectation value of the electromagnetic field $\langle \hat{A}_{\mu}(x) \rangle$ is zero in quantum electrodynamics, it does not vanish in our generalized theory with arbitrary propagators S and D [see Eq. (4.6)].

The derivative of vacuum diagrams with respect to the electron kernel S^{-1},

$$ {}^{e}G^{2}_{12} = \frac{\delta}{\delta S^{-1}_{21}} W[S^{-1}, D^{-1}, V], \tag{5.7} $$

leads to the electronic two-point function, whose diagrams are



(5.8)

B. Scattering processes

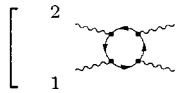
The generation of diagrams for scattering processes between electrons and photons (5.2) and higher even-point functions is now straightforward.

1. Photon-photon scattering

The four-point function of photons is obtained by cutting two photon lines in the vacuum diagrams or one photon line in the photonic two-point function:

$$ {}^{\gamma\gamma}G^{4}_{1234} = 4 \left\{ \frac{\delta^2 W}{\delta D^{-1}_{12} \delta D^{-1}_{34}} + \frac{\delta W}{\delta D^{-1}_{12}} \frac{\delta W}{\delta D^{-1}_{34}} \right\} = -2 \frac{\delta\, {}^{\gamma}G^{2}_{12}}{\delta D^{-1}_{34}} + {}^{\gamma}G^{2}_{12}\, {}^{\gamma}G^{2}_{34}. \tag{5.9} $$

After applying one of the two possible operations in Eq. (5.9), the resulting connected diagrams to order e^4 are

$$ {}^{\gamma\gamma}G^{4,c}_{1234} \equiv -e^4 \left[\right. $$


$$ + \ 5 \text{ perm.} \left. \right] + \mathcal{O}(e^6), \tag{5.10} $$

each permutation of two external spacetime coordinates leading to a different diagram.

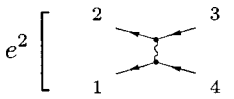
2. Møller and Bhabba scattering

The scattering of two electrons (Møller scattering) is described by the electronic four-point function

$$ {}^{ee}G^{4}_{1234} = \frac{\delta^2 W}{\delta S^{-1}_{41} \delta S^{-1}_{32}} + \frac{\delta W}{\delta S^{-1}_{41}} \frac{\delta W}{\delta S^{-1}_{32}} = \frac{\delta\, {}^{e}G^{2}_{23}}{\delta S^{-1}_{41}} + {}^{e}G^{2}_{14}\, {}^{e}G^{2}_{23}. \tag{5.11} $$

To order e^4, the connected diagrams contributing to the fermionic four-point function are

$$^{ee}G^{4,c}_{1234} \equiv e^2 \Big[\quad - (3 \leftrightarrow 4) \Big] + e^4 \Big[\quad + \quad + \quad + \quad + \quad + \quad - \quad - \quad - \quad - \quad - (3 \leftrightarrow 4) \Big] + \mathcal{O}(e^6), \tag{5.12}$$



where the spacetime indices in all diagrams are arranged as in the first. Each diagram on the right-hand side has a partner with opposite sign, where the spacetime indices either of the incoming or of the outgoing electrons are interchanged. The tadpole diagrams vanish for physical propagators $S=S_F$, $D=D_F$, and the corresponding corrections attached to external legs do not contribute when calculating the S-matrix elements. In our general vacuum functional, however, we must not discard them, since they contribute to higher functional derivatives, which would be needed for the calculation of, e.g., the six-point function.

By interchanging spacetime arguments in the kernels of Eq. (5.12) apparently, the Feynman diagrams (5.12) describe also scattering of electron and positron (Bhabba scattering) and scattering of two positrons.

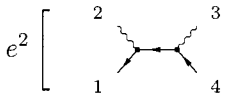
3. Compton scattering

The amplitude of Compton scattering is given by the mixed four-point function $^{e\gamma}G^4_{1234}$. To obtain the relevant Feynman diagrams, we have to perform one of the possible operations

$$\begin{aligned} {}^{e\gamma}G^4_{1234} &= -2\left\{ \frac{\delta^2 W}{\delta D^{-1}_{23}\,\delta S^{-1}_{41}} + \frac{\delta W}{\delta D^{-1}_{23}} \frac{\delta W}{\delta S^{-1}_{41}} \right\} \\ &= \frac{\delta\, {}^{\gamma}G^2_{23}}{\delta S^{-1}_{41}} + {}^{e}G^2_{14}\, {}^{\gamma}G^2_{23} = -2\,\frac{\delta\, {}^{e}G^2_{14}}{\delta D^{-1}_{23}} + {}^{e}G^2_{14}\, {}^{\gamma}G^2_{23}. \end{aligned} \tag{5.13}$$

The resulting connected Feynman diagrams to order e^4 are

$$^{e\gamma}G^{4,c}_{1234} \equiv e^2 \Big[\quad + (2 \leftrightarrow 3) \Big] + e^4 \Big[\quad + \quad + \quad + \quad + \quad + \quad - \quad - \quad - \quad - \quad - \quad + (2 \leftrightarrow 3) \Big] + \mathcal{O}(e^6), \tag{5.14}$$



where the diagrams with interchanged photon coordinates $2 \leftrightarrow 3$ possess the same sign as the original one.

C. Three-point vertex function

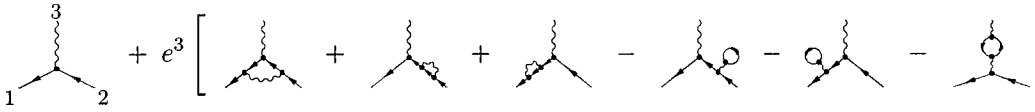
The three-point vertex function is obtained from the vacuum energy W by performing the derivative with respect to the interaction function V_{123}, which we have defined in Eq. (3.32):

$$G^3_{123} = -\frac{1}{e}\,\frac{\delta W}{\delta V_{213}}. \tag{5.15}$$

The easiest way to find the associated Feynman diagrams is to apply the graphical operation (3.33), which removes a vertex from the vacuum diagrams in all possible ways and lets the remaining legs open. Dropping disconnected diagrams by considering the cumulant

$$G^{3,c}_{123}=G^{3}_{123}-\langle\hat{\psi}_1\hat{\overline{\psi}}_2\rangle\langle\hat{A}_3\rangle \tag{5.16}$$

we obtain

$G^{3,c}_{123} =$  $+\mathcal{O}(e^5).$

(5.17)

VI. SCATTERING OF ELECTRONS AND PHOTONS IN THE PRESENCE OF AN EXTERNAL ELECTROMAGNETIC FIELD

To describe the scattering of electrons and photons on external electromagnetic fields, the action $\mathcal{A}[\overline{\psi},\psi,A]$ in Eq. (2.13) must be extended by an additional external current J, which is coupled linearly to the electromagnetic field A:

$$\mathcal{A}^J[\overline{\psi},\psi,A,J]=\mathcal{A}[\overline{\psi},\psi,A]-e\int_1 J_1A_1. \tag{6.1}$$

Then the partition function (2.12) becomes a functional in the physical current J and is given by

$$Z[J]=\oint\mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}Ae^{-\mathcal{A}^J[\overline{\psi},\psi,A,J]} \tag{6.2}$$

with $Z=Z[0]$. The external current is usually supplied by some atomic nucleus of charge Ne with integer number N. For this reason, the factor e is removed from the current in Eq. (6.1) to be able to collect systematically all Feynman diagrams of the same order in e. This organization may not always be the most useful one. If we consider, for instance, an external heavy nucleus with a high charge Ne, we may have to include many more orders in the external charge Ne than in the internal charge e. Such subleties will be ignored here, for simplicity.

A. Recursion relation for the vacuum energy with external source

Along similar lines as before, we derive the recursion relation for the vacuum energy in the presence of an external current, $W[J]=\ln Z[J]$ which is now also a functional of J (suppressing the other arguments D^{-1},S^{-1},V). After that, we derive a recursion relation only producing those vacuum diagrams which contain a coupling to the source. It turns out that the resulting recursion relation for current diagrams is extremely simple. Hence, this recursion relation is the ideal extension of the former Eq. (4.16) which generates only the source-free diagrams.

1. Complete recursion relation for all vacuum diagrams

The recursion relation for all vacuum diagrams with and without external source is derived in a similar manner as that for all source-free vacuum diagrams (4.16). There will be, however, a few significant differences in comparison with the procedure in Sec. IV. Since the current J couples to the electromagnetic field A, vacuum diagrams with external current always contain photon lines. For this reason, we start with the identity

$$\oint\mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A\frac{\delta}{\delta A_1}\{A_2e^{-\mathcal{A}^J[\overline{\psi},\psi,A,J]}\}=0 \tag{6.3}$$

instead of Eq. (4.1). Performing the functional derivative leads to

$$Z[J]\delta_{12}+2\int_3 D^{-1}_{13}\frac{\delta Z[J]}{\delta D^{-1}_{23}}-e\int_{34}V_{341}\frac{\delta}{\delta S^{-1}_{34}}[\langle\hat{A}_2\rangle^JZ[J]]$$

$$+eJ_1\langle\hat{A}_2\rangle^JZ[J]=0 \tag{6.4}$$

in analogy to Eq. (4.3). The expectation value of the electromagnetic field A in the presence of an external source J is found by exploiting the identity

$$\oint\mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A\frac{\delta}{\delta A_1}e^{-\mathcal{A}^J[\overline{\psi},\psi,A,J]}=0 \tag{6.5}$$

which is used to derive, as in Eqs. (4.4)–(4.6),

$$\langle\hat{A}_1\rangle^J=-e\int_{234}V_{234}D_{14}\frac{\delta W[J]}{\delta S^{-1}_{23}}+e\int_2 D_{12}J_2, \tag{6.6}$$

where we have set $W[J]=\ln Z[J]$. Inserting the expectation value (6.6) into Eq. (6.4), the resulting functional differential equation reads

$$\delta_{12}+2\int_3 D^{-1}_{13}\frac{\delta W[J]}{\delta D^{-1}_{23}}=-e^2\int_{3\cdots7}V_{341}V_{567}D_{27}\left\{\frac{\delta^2W[J]}{\delta S^{-1}_{34}\delta S^{-1}_{56}}+\frac{\delta W[J]}{\delta S^{-1}_{34}}\frac{\delta W[J]}{\delta S^{-1}_{56}}\right\}$$

$$+2e^2\int_{345}V_{345}D_{25}J_1\frac{\delta W[J]}{\delta S^{-1}_{34}}-e^2\int_3 J_1D_{23}J_3. \tag{6.7}$$

Using relations (3.11) and (3.30), and taking the trace, this becomes

$$\int_1 \delta_{11} + 2\int_{12} D_{12}\frac{\delta W[J]}{\delta D_{12}} = 2e^2\int_{1\cdots 8} V_{123}V_{456}D_{36}S_{71}S_{24}S_{58}\frac{\delta W[J]}{\delta S_{78}} + e^2\int_{1\cdots 10} V_{123}V_{456}D_{36}S_{71}S_{28}S_{94}S_{510}$$
$$\times\left\{\frac{\delta^2 W[J]}{\delta S_{78}\delta S_{910}} + \frac{\delta W[J]}{\delta S_{78}}\frac{\delta W[J]}{\delta S_{910}}\right\} + 2e^2\int_{1\cdots 6} V_{123}D_{34}J_4S_{51}S_{26}\frac{\delta W[J]}{\delta S_{56}} + e^2\int_{12} J_1 D_{12}J_2, \quad (6.8)$$

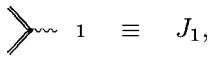
which generalizes Eq. (4.8). Expanding $W[J]$ as in Eq. (2.26),

$$W[J] = W^{(0)} + \sum_{p=1}^{\infty} e^{2p} W^{(p)}[J], \quad (6.9)$$

and using the fact that the free vacuum energy $W^{(0)}[J] = W^{(0)}[0] = W^{(0)}$ is independent of the external current, the first term on the left-hand side in Eq. (6.8) is canceled by an identity following from Eq. (3.2)

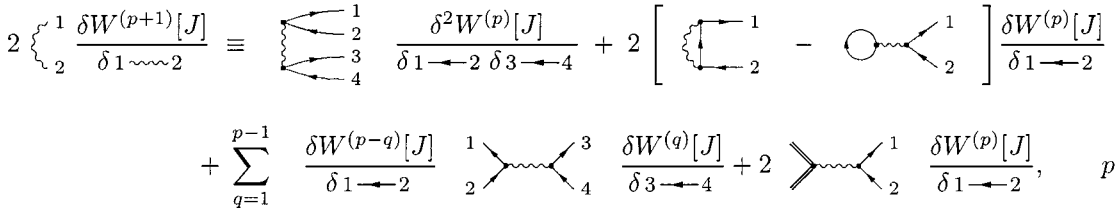
$$2\int_{12} D_{12}\frac{\delta W^{(0)}}{\delta D_{12}} = \int_1 \delta_{11}. \quad (6.10)$$

Introducing a Feynman diagram for the coupling to the current J



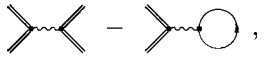
$$\equiv \quad J_1, \quad (6.11)$$

we obtain the graphical recursion relation



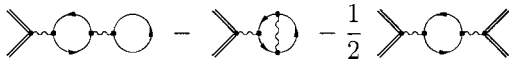
(6.12)

and the first-order diagrams



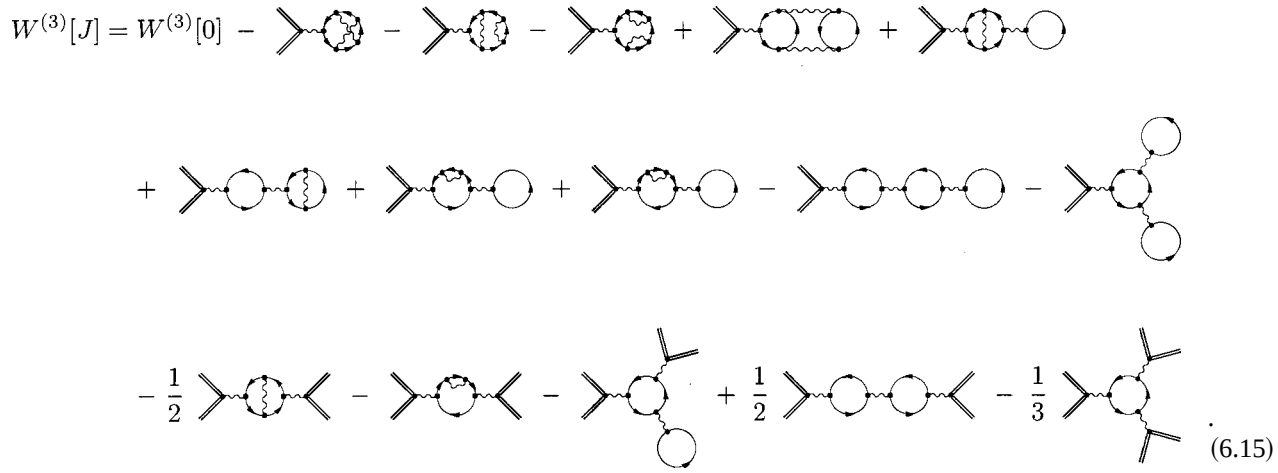
(6.13)

where $W^{(1)}[0] = W^{(1)}$ contains the source-free first-order vacuum diagrams (3.41). An important difference between the recursion relation (6.12) and the previous (4.16) is that the vacuum diagrams in a series of the coupling constant e contain different numbers of photon (or electron) lines, thus not satisfying a simple eigenvalue equation like (4.11). In fact, each vacuum diagram, generated by using the right-hand side of the recursion relation (6.12), must be divided by twice the number of photon lines in the diagram to obtain the correct weight factor. This procedure is a consequence of the left-hand side of Eq. (6.12), which counts the number of photon lines in each diagram separately. By taking this into consideration, the second-order vacuum diagrams are given by



(6.14)

with the source-free diagrams given in Eq. (4.18). In third order, there are 15 diagrams which couple to the physical source:

$W^{(3)}[J] = W^{(3)}[0]$  (6.15)

In the following we derive a recursion relation which allows us to generate only those vacuum diagrams which contain a coupling to the source.

2. Recursion relation for vacuum diagrams coupled to the external source

Since we have the possibility to generate all source-free vacuum diagrams with the help of the recursion relation (4.16), we are able to set up a recursion relation to generate only the diagrams with source coupling. Inserting on the left-hand side of Eq. (6.6) the equation

$$\langle \hat{A}_1 \rangle^J = \frac{1}{e} \frac{\delta W[J]}{\delta J_1}, \tag{6.16}$$

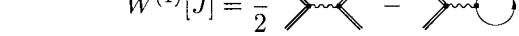
multiplying both sides with J_1, and performing the integral $\int_1$ yields

$$\int_1 J_1 \frac{\delta W[J]}{\delta J_1} = e^2 \int_{1\cdots 6} V_{234} D_{14} J_1 S_{52} S_{36} \frac{\delta W[J]}{\delta S_{56}} + e^2 \int_{12} J_1 D_{12} J_2 . \tag{6.17}$$

On the right-hand side we have changed the functional derivatives with respect to the kernel S^{-1} into functional derivatives with respect to the propagator S using Eq. (3.30). Inserting the decomposition (6.9) and utilizing the fact that $W^{(0)}$ from Eq. (2.22) is source-free, $\delta W^{(0)}/\delta J_1 = 0$, we find

$$\int_1 J_1 \frac{\delta W^{(1)}[J]}{\delta J_1} + \sum_{n=1}^{\infty} e^{2n} \int_1 J_1 \frac{\delta W^{(n+1)}[J]}{\delta J_1} = -\int_{1\cdots 4} V_{234} D_{14} J_1 S_{32} + \int_{12} J_1 D_{12} J_2 + \sum_{n=1}^{\infty} e^{2n} \int_{1\cdots 6} V_{234} D_{14} J_1 S_{52} S_{36} \frac{\delta W^{(n)}[J]}{\delta S_{56}} . \tag{6.18}$$

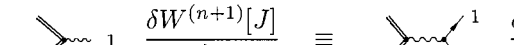
To lowest order, the right-hand side yields the source diagrams

$\tilde{W}^{(1)}[J] =$  (6.19)

where we have used the wiggle to indicate the restriction to the source diagrams of $W^{(1)}[J]$ in Eq. (6.13). The full functional solving Eq. (6.18) consists of the terms

$$W^{(n)}[J] = W^{(n)}[0] + \tilde{W}^{(n)}[J], \tag{6.20}$$

where the source-free contributions $W^{(n)}[0] = W^{(n)}$ of Sec. IV represent integration constants undetermined by Eq. (6.18). Introducing a diagram for the functional derivative with respect to the current J,

 (6.21)

the recursion relation for the vacuum diagrams with source-coupling Eq. (6.18) is graphically written for $n \geq 1$ as

 (6.22)

The graphical operation on the right-hand side means that an external current is attached through a photon line to a fermion line in all possible ways. The iteration of this recursion relation is very simple since the right-hand side is linear. Each diagram calculated with the right-hand side of this equation must be divided by the number of source-coupling within the diagram since the operation on the left-hand side counts the number of source-couplings in the diagram. By considering Eq. (6.20), one easily reproduces the higher-order vacuum diagrams given in the Eqs. (6.14) and (6.15).

B. Scattering of electrons and photons in the presence of an external source

Typically, an external electromagnetic field is produced by a heavy particle such as a nucleus or an ion. Quantum electrodynamical effects like pair creation, Bremsstrahlung, and Lamb shift are caused by such electromagnetic fields. The Feynman diagrams for the *n*-point functions associated with these processes are again obtained by cutting electron or photon lines from the just-derived vacuum diagrams.

1. Vacuum polarization induced by external field

The photon propagator in the presence of an external source

$$ {}^{\gamma}G^{2}_{12}[J] = -2\,\frac{\delta W[J]}{\delta D^{-1}_{12}} \tag{6.23}$$

is found by cutting a photon line in the vacuum diagrams (6.13)–(6.15):

$$ {}^{\gamma}G^{2,c}_{12}[J] = {}^{\gamma}G^{2,c}_{12}[0] + e^{4}\Big[- \quad \overset{1}{\underset{2}{\text{⟩◯⟨}}} \quad - (1 \leftrightarrow 2)\Big] + \mathcal{O}(e^{6}), \tag{6.24}$$

showing polarization caused by the external field.

2. Lamb-shift and anomalous magnetic moment

The important phenomena of Lamb shift and anomalous magnetic moments are obtained from the perturbative corrections in the electron propagator:

$$ {}^{e}G^{2}_{12}[J] = \frac{\delta W[J]}{\delta S^{-1}_{21}}, \tag{6.25}$$

whose diagrams come from cutting an electron line in the vacuum diagrams (6.13)–(6.15). To order e^4, we have

$$ {}^{e}G^{2}_{12}[J] = {}^{e}G^{2}_{12}[0] + e^{2}\ \overset{1}{\underset{2}{\text{⟩⟨}}} + e^{4}\Big[\quad + \quad + \quad + \quad - \quad - \quad - \quad + \quad + \quad \Big] + \mathcal{O}(e^{6}). \tag{6.26}$$

As already mentioned before, diagrams with corrections on external legs and tadpole graphs do not contribute to *S*-matrix elements. In some problems, diagrams with more than one source-coupling are irrelevant.

3. Pair creation, pair annihilation and Bremsstrahlung

By differentiating the vacuum energy diagrams (6.13)–(6.15) with respect to the interaction function V_{123}, we obtain the vertex function in the presence of an external field:

$$ G^{3}_{123}[J] = -\frac{1}{e}\,\frac{\delta W[J]}{\delta V_{213}}. \tag{6.27}$$

The connected Feynman diagrams are to order e^3:

$$ G^{3,c}_{123}[J] = G^{3,c}_{123}[0] + e^{3}\Big[\quad \overset{3}{\underset{1}{\text{⟩}}}\!\!\underset{2}{\text{⟨}} \quad + \quad \overset{3}{\underset{1}{\text{⟨}}}\!\!\underset{2}{\text{⟩}} \quad \Big] + \mathcal{O}(e^{5}) \tag{6.28}$$

with $G^{3,c}_{123}[0] = G^{3,c}_{123}$ of Eq. (5.17). These diagrams appear in pair creation, pair annihilation, or Bremsstrahlung processes.

VII. SUMMARY

We have introduced a graphical recursion relation obeyed by the vacuum diagrams in quantum electrodynamics based on functional analytic methods developed in Refs. [6], [7]. Its iterative solution allows us to generate all vacuum diagrams with their correct weights order by order in perturbation theory by removing and joining lines. By removing photon and electron lines as well as vertices from the vacuum graphs, we obtain all diagrams of scattering processes with their multiplicities. The method also generates all diagrams of processes involving external sources.

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