Nonperturbative XY-model approach to strong coupling superconductivity in two and three dimensions

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For an electron gas with δ-function attraction, we investigate the crossover from weak- to strong-coupling superconductivity in two and three dimensions. From mean-field theory we extract the stiffness of phase fluctuations and set up effective XY models which serve to determine nonperturbatively the temperature of phase decoherence where superconductivity breaks down. We find the transition temperature T_c as a monotonous function of the coupling strength both in two and three dimensions, and give analytic formulas for the merging of temperature of phase decoherence with the temperature of pair formation in the weak-coupling limit. [S0163-1829(99)07605-5]

I. INTRODUCTION

The crossover from BCS superconductors to a Bose-Einstein condensate of tightly bound fermion pairs was first studied many years ago in Refs. 1–3 in a model with δ-function attraction. This crossover has recently raised renewed interest,[4–34] especially after the publication of experimental results on cuprate superconductors and their theoretical interpretation in Ref. 33. In this paper we present a detailed study of the crossover based on a nonperturbative procedure in which mean-field properties are used to set up an effective XY model whose well-known nonperturbative properties render information on the entire crossover regime of the above model.

Physically, the most important distinctions between conventional weak-coupling (BCS) and strong-coupling (Bose-Einstein) regime lies in the fact that in the former only a small fraction of the conduction electrons is paired with the superfluid density involving all pairs, whereas in the latter practically all carriers are paired below a certain temperature T^*, although not condensed. The temperature has to be lowered further below some critical temperature $T_c < T^*$ to make these pairs condense and establish phase coherence, which leads to superconductive behavior. We shall neglect the coupling to the magnetic vector potential throughout the forthcoming discussion, so that the phase coherence below T_c can be of long range, unspoiled by the Meissner effect which would reduce the range to a finite penetration depth. In the model to be investigated in this paper, the crossover from BCS- to Bose-type superconductivity will take place either by varying the coupling strength, or by decreasing the carrier density.

Since in the BCS theory pair binding is weak, it can well be described by mean-field theory for the pair fields. In the opposite limit of strong pair bonding, on the other hand, superconductivity sets in via a macroscopic occupation of the $\mathbf{q}=\mathbf{0}$ level, and we are obliged to go beyond mean-field theory to describe it. In three dimensions, crossover from BCS superconductivity to the Bose-Einstein condensation of tightly bound fermion pairs was first investigated in Ref. 2 by summing particle-particle ladder diagrams which correspond to Gaussian fluctuations around the mean field. In the functional integral formalism this was studied in Ref. 5. In both papers, fluctuation corrections were retained in the number equation, which was solved together with the mean-field gap equation. In this approximation, starting from a fermionic system, the gas of electron pairs was mapped in the strong-coupling limit to an ideal Bose gas, and the critical temperature asymptotically becomes the temperature of the Bose-Einstein condensation of an ideal Bose gas of particles with mass $2m$ and density $n/2$, where m and n are the fermion mass and density. The critical temperature T_c has an artificial maximum at an intermediate coupling strength, thus approaching a limiting value in the strong-coupling limit from above. This artifact was removed in the generalized self-consistent Green-function numerical approach in Ref. 9.

In this paper we shall study the properties of collective modes with help of the lowest gradient terms governing the Gaussian fluctuations around the mean-field solution. These fluctuations are most violent in the phase of the order parameter. Phase transitions in a system with these fluctuations are well understood in two and three dimensions from extensive studies of the XY model. By setting up an equivalent XY model we are therefore able to describe very well the onset and disappearance of superconductivity in the entire crossover regime. In this way, we shall obtain simple formulas for the critical temperature T_c, which turns out to be a monotonously increasing function of the coupling strength and carrier density in both two and three dimensions. In the weak-coupling limit, we give simple explicit formulas which show how the temperature of the XY-model transition converges to the transition temperature in the BCS theory.

II. MODEL

The weak- to strong-coupling crossover in two dimensions was studied via the Kosterlitz-Thouless theory in Refs.

26 and 24. In Ref. 24 it was investigated within the same model as ours at a fixed carrier density, but only numerically. We shall see in Sec. IV that these numerical results do not cover the entire crossover region; in particular the merging of T_{KT} and T^* in the weak-coupling region is missing—the phenomenon that we describe analytically in our paper. The different properties of size and phase fluctuations were also exploited in Ref. 36.

The Hamiltonian of our model is the typical BCS Hamiltonian in D dimensions ($\hbar=1$),

$$H=\sum_{\sigma}\int d^Dx\,\psi_\sigma^\dagger(\mathbf{x})\left(-\frac{\nabla^2}{2m}-\mu\right)\psi_\sigma(\mathbf{x}) +g\int d^Dx\,\psi_\uparrow^\dagger(\mathbf{x})\psi_\downarrow^\dagger(\mathbf{x})\psi_\downarrow(\mathbf{x})\psi_\uparrow(\mathbf{x}), \tag{1}$$

where $\psi_\sigma(\mathbf{x})$ is the Fermi field operator, $\sigma=\uparrow,\downarrow$ denotes the spin components, m is the effective fermionic mass, and $g<0$ the strength of an attractive potential $g\,\delta(\mathbf{x}-\mathbf{x}')$.

The mean-field equations for the gap parameter Δ and the chemical potential μ are obtained in the standard way from the equations (see, for example, Ref. 35)

$$-\frac{1}{g}=\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{2E_{\mathbf{k}}}\tanh\frac{E_{\mathbf{k}}}{2T}, \tag{2}$$

$$n=\frac{N}{V}=\frac{1}{V}\sum_{\mathbf{k}}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\frac{E_{\mathbf{k}}}{2T}\right), \tag{3}$$

where the sum runs over all wave vectors $\mathbf{k}$, N is the total number of fermions, V the volume of the system, and

$$E_{\mathbf{k}}=\sqrt{\xi_{\mathbf{k}}^2+\Delta^2}\qquad\text{with}\qquad \xi_{\mathbf{k}}=\frac{\mathbf{k}^2}{2m}-\mu \tag{4}$$

are the energies of single-particle excitations.

The δ-function potential produces an artificial divergence, and requires regularization. A BCS superconductor possesses a natural cutoff supplied by the Debye frequency ω_D. For the crossover problem to be treated here this is no longer a useful quantity, since in the strong-coupling limit all fermions participate in the interaction, not only those in a thin shell of width ω_D around the Fermi surface. To be applicable in this regime, we renormalize the gap equation in three dimensions with the help of the experimentally observable s-wave scattering length a_s, for which the low-energy limit of the two-body scattering process gives an equally divergent expression[9–13]

$$\frac{m}{4\pi a_s}=\frac{1}{g}+\frac{1}{V}\sum_{\mathbf{k}}\frac{m}{\mathbf{k}^2}. \tag{5}$$

Eliminating g from Eqs. (5) and (2), we obtain a renormalized gap equation

$$-\frac{m}{4\pi a_s}=\frac{1}{V}\sum_{\mathbf{k}}\left[\frac{1}{2E_{\mathbf{k}}}\tanh\frac{E_{\mathbf{k}}}{2T}-\frac{m}{\mathbf{k}^2}\right], \tag{6}$$

in which $1/k_Fa_s$ plays the role of a dimensionless coupling constant which monotonically increases from $-\infty$ to ∞ as the bare coupling constant g runs from small (BCS limit) to large values (BE limit). This equation is to be solved simultaneously with Eq. (3). These mean-field equations were analyzed at a fixed carrier density in Refs. 5 and 7. Here we shall first reproduce some of the earlier estimates for T^* and μ.

In the BCS limit, the chemical potential μ does not differ much from the Fermi energy ϵ_F, whereas with increasing interaction strength the distribution function $n_{\mathbf{k}}$ broadens and μ decreases. In the BE limit we have tightly bound pairs and nondegenerate fermions with a large negative chemical potential, $|\mu|\gg T$. In the strong-coupling limit, Eq. (6) provides us with an estimate for T^*, the characteristic temperature of the thermal pair breaking,[5] whereas Eq. (3) determines μ. From Eq. (3) we obtain in the BE limit $\mu=-E_b/2$, where $E_b=1/ma_s^2$ is the binding energy of the bound pairs. In the BE limit, we can estimate that the pseudogap sets in at $T^*\simeq E_b/2\ln(E_b/\epsilon_F)^{3/2}$. A simple ''chemical'' equilibrium estimate ($\mu_b=2\mu_f$) yields $T_{\rm dissoc}\simeq E_b/\ln(E_b/\epsilon_F)^{3/2}$ for the temperature of pair dissociation, which shows that at strong couplings T^* is indeed related to pair formation.[5,6]

The gap in the spectrum of single-particle excitations has a special feature[3,1,7] when the chemical potential changes its sign. The sign change occurs at the minimum of the Bogoliubov quasiparticle energy $E_{\mathbf{k}}$, which defines the gap energy in the quasiparticle spectrum:

$$E_{\rm gap}=\min(\xi_{\mathbf{k}}^2+\Delta^2)^{1/2}. \tag{7}$$

Thus, for a positive chemical potential, the gap energy is given directly by the gap function Δ, whereas for negative chemical potential, it is larger than that:

$$E_{\rm gap}=\begin{cases}\Delta & \text{for } \mu>0,\\ (\mu^2+\Delta^2)^{1/2} & \text{for } \mu<0.\end{cases} \tag{8}$$

In two dimensions, a nonzero bound-state energy ϵ_0 exists for any coupling strength. The cutoff can therefore be eliminated by subtracting, from the two-dimensional zero-temperature gap equation[11–13]

$$-\frac{1}{g}=\frac{1}{2V}\sum_{\mathbf{k}}\frac{1}{\sqrt{\xi_{\mathbf{k}}^2+\Delta^2}}=\frac{m}{4\pi}\int_{-x_0}^{\infty}dz\,\frac{1}{\sqrt{1+z^2}}, \tag{9}$$

where $z=k^2/2m\Delta-x_0$, $x_0=\Delta/\mu$, the bound-state equation

$$-\frac{1}{g}=\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{\mathbf{k}^2/m+\epsilon_0}=\frac{m}{2\pi}\int_{-x_0}^{\infty}dz\,\frac{1}{2z+\epsilon_0/\Delta+2x_0}. \tag{10}$$

After performing the elementary integrals, we find

$$\frac{\epsilon_0}{\Delta}=\sqrt{1+x_0^2}-x_0. \tag{11}$$

In the next sections we will work at finite temperature; in doing so, we do not fix the carrier density but assume the presence of a reservoir which provides us with a temperature-independent chemical potential $\mu=\mu(1/k_Fa_s;T=0)$. Such a fixed μ will be most convenient for deriving simple analytic results for the finite-temperature behavior of the system. In this fixed-μ model, the carrier density becomes temperature dependent. In Ref. 24, the tem-

perature dependence of the chemical potential was calculated numerically for the entire crossover region within such a ''fixed carrier density model,'' where it turned out to be very small in comparison with the dependence on the coupling strength. For experimental measurements of μ see Ref. 37.

III. PHASE FLUCTUATIONS IN TWO DIMENSIONS AND KOSTERLITZ-THOULESS TRANSITION

In this section we make use of derivative expansion to determine the relevant stiffness parameter for the study of phase fluctuations, which in two dimensions determines the temperature of the Kosterlitz-Thouless (KT) transition. In a two-dimensional system, the phase fluctuations are most violent causing the strongest modifications of the mean-field properties. The Coleman-Mermin-Wagner-Hohenberg theorem[38] forbids the existence of a strict long-range order, but there is quasi-long-range order manifesting itself in a power behavior of the correlation functions at all temperatures below T_{KT}.

The effective Hamiltonian from which we deduce the stiffness of the phase fluctuations was derived in Refs. 24 and 41. In this section we summarize a few important aspects of it, with a reminder of its derivation given below. Writing the space-time-dependent order parameter as $\Delta(x)e^{i\theta(x)}$, where x denotes the 4-vector $x=(\tau,\mathbf{x})$ formed from imaginary time and position vector, the partition function may be written as a functional integral[41,39,40]

$$Z(\mu,T)=\int \Delta\,\mathcal{D}\Delta\,\mathcal{D}\theta\exp\{-\beta\Omega[\mu,T,\Delta(x),\partial\theta(x)]\}, \tag{12}$$

where

$$\beta\Omega[\mu,T,\Delta(x),\partial\theta(x)] =\frac{1}{g}\int_0^\beta d\tau\int d\mathbf{x}\,\Delta^2(x)-\mathrm{Tr}\ln G^{-1}+\mathrm{Tr}\ln G_0^{-1} \tag{13}$$

is the one-loop effective action, containing the inverse Green function of the fermions in the collective pair field

$$G^{-1}=-\hat{I}\partial_\tau+\tau_3\left(\frac{\nabla^2}{2m}+\mu\right)+\tau_1\Delta(\tau,\mathbf{x}) -\tau_3\left[\frac{i\partial_\tau\theta(\tau,\mathbf{x})}{2}+\frac{[\nabla\theta(\tau,\mathbf{x})]^2}{8m}\right] +\hat{I}\left[\frac{i\nabla^2\theta(\tau,\mathbf{x})}{4m}+\frac{i\nabla\theta(\tau,\mathbf{x})\nabla}{2m}\right]. \tag{14}$$

Here τ_1 and τ_3 are the usual Pauli matrices, and $G_0=G|_{\mu,\Delta,\theta=0}$ is added for regularization.

Let us now assume that phase gradients are small. Then $\Omega[\mu,T,\Delta(x),\partial\theta(x)]$ can be approximated as follows:

$$\Omega[\mu,\Delta(x),\partial\theta(x)]\simeq\Omega_{\mathrm{kin}}[\mu,T,\Delta,\partial\theta(x)]+\Omega_{\mathrm{pot}}(\mu,T,\Delta), \tag{15}$$

with the ''kinetic'' term (see Refs. 41 and 40)

$$\Omega_{\mathrm{kin}}[\mu,T,\Delta,\partial\theta(x)]=T\,\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}(\mathcal{G}\Sigma)^n|_{\Delta=\mathrm{const}}, \tag{16}$$

and the ''potential'' term

$$\Omega_{\mathrm{pot}}(\mu,T,\Delta)=\left(\frac{1}{g}\int d^Dx\,\Delta^2-T\,\mathrm{Tr}\ln\mathcal{G}^{-1}+T\,\mathrm{Tr}\ln G_0^{-1}\right)\Bigg|_{\Delta=\mathrm{const}}. \tag{17}$$

The latter coincides with the mean-field energy, determining the modulus of $\Delta(\mu,T)$. The kinetic part Ω_{kin} contains gradient terms governing the stiffness of the phase fluctuations whose size is determined by the modulus of $\Delta(\mu,T)$. Both Ω_{kin} and Ω_{pot} are expressed in terms of the Green function of the fermions, which solves the equation

$$\left[-\hat{I}\partial_\tau+\tau_3\left(\frac{\nabla^2}{2m}+\mu\right)+\tau_1\Delta\right]\mathcal{G}(\tau,\mathbf{x}) =\delta(\tau)\,\delta(\mathbf{x}). \tag{18}$$

The operator $\Sigma(\partial\theta)$ in Eq. (16) is

$$\Sigma(\partial\theta)\equiv\tau_3\left[\frac{i\partial_\tau\theta}{2}+\frac{(\nabla\theta)^2}{8m}\right]-\hat{I}\left[\frac{i\nabla^2\theta}{4m}+\frac{i\nabla\theta(\tau,\mathbf{x})\nabla}{2m}\right]. \tag{19}$$

The gradient expansion that we use to determine stiffness was first performed in Ref. 41 at zero temperature. In Ref. 24, the kinetic term Ω_{kin} was calculated in two dimensions at finite temperature for arbitrary chemical potential retaining terms with $n=1,2$ in expansion (16).

The result is

$$\Omega_{\mathrm{kin}}=\frac{T}{2}\int_0^\beta d\tau\int d^Dx\{n(\mu,T,\Delta)i\partial_\tau\theta +J[\mu,T,\Delta(\mu,T)](\nabla\theta)^2 +K[\mu,T,\Delta(\mu,T)](\partial_\tau\theta)^2\}, \tag{20}$$

where $J(\mu,T,\Delta)$ is the stiffness coefficient, whose explicit form is

$$J(\mu,T,\Delta)=\frac{1}{4m}n(\mu,T,\Delta) -\frac{T}{4\pi}\int_{-\mu/2T}^{\infty}dx\frac{x+\mu/2T}{\cosh^2\sqrt{x^2+\Delta^2/4T^2}}. \tag{21}$$

The other coefficients are

$$K(\mu,T,\Delta)=\frac{m}{8\pi}\left(1+\frac{\mu}{\sqrt{\mu^2+\Delta^2}}\tanh\frac{\sqrt{\mu^2+\Delta^2}}{2T}\right), \tag{22}$$

and $n(\mu,T,\Delta)$, the density of fermions (3) which varies with temperature in our model. At the temperature T^* where the modulus of Δ vanishes, the stiffness also disappears.

We are now ready to set up an effective XY model governing the phase fluctuations. The model Hamiltonian corresponding to the gradient term in $\theta(x)$ is[42,8]

$$H=\frac{J}{2}\int d\mathbf{x}[\nabla\theta(\mathbf{x})]^2. \tag{23}$$

In contrast to the standard XY model, the stiffness parameter is not a constant but depends on temperature via the solution of gap and number equations (2) and (3).

In this model, a Kosterlitz-Thouless vortex unbending transition takes place below T^*. For vortices of a high fugacity, the temperature of the phase transition is determined by the well-known formula[43]

$$T_{\rm KT}=\frac{\pi}{2}J, \tag{24}$$

which follows from the divergence of the average square size of a vortex-antivortex pair. Since these attract each other by a Coulomb potential $v(r)=2\pi J\ln(r/r_0)$, the average square distance is

$$\langle r^2\rangle\propto\int_{r_0}^{\infty}dr\,r\,r^2e^{-(2\pi J/T)\ln(r/r_0)}$$

$$\propto\frac{1}{4-2\pi J/T}, \tag{25}$$

which diverges indeed at the temperature (24). In our case $T_{\rm KT}$ should be determined self-consistently:

$$T_{\rm KT}=\frac{\pi}{2}J[\mu,T_{\rm KT},\Delta(\mu,T_{\rm KT})]. \tag{26}$$

From Eqs. (21), (3), and (24), it is easily seen that $T_{\rm KT}$ indeed tends to zero when the pair attraction vanishes, in which case $\Delta(T=0)=0$. In general, the behavior of $T_{\rm KT}$ for strong and weak couplings is found by the following considerations. We observe that the particle number n does not vary appreciably in these limits with temperature in the range $0<T<T^*$, so that weak-coupling estimates for $T_{\rm KT}$ derived within the model with a temperature-independent chemical potential (i.e., when the system is coupled to a large reservoir) practically coincide with those derived from a fixed fermion density. Further it is immediately realized that in the weak-coupling limit, $\Delta(T_{\rm KT},\mu)/T_{\rm KT}$ is a small parameter. At zero coupling, the stiffness $J[\mu,T_{\rm KT},\Delta(\mu,T_{\rm KT})]$ vanishes identically, such that an estimate of J at weak couplings requires calculating a lowest-order correction to the second term of Eq. (21) proportional to $\Delta(T_{\rm KT},\mu)/T_{\rm KT}$. Thus the weak-coupling expression for stiffness reads

$$J(T)\simeq\frac{7\zeta(3)}{16\pi^3}\epsilon_F\frac{\Delta^2(T)}{T^{*2}}. \tag{27}$$

Equating this with the stiffness in Eq. (24), we obtain the weak-coupling equation for $T_{\rm KT}$:

$$T_{\rm KT}\simeq\frac{\epsilon_F}{4}\left(1-\frac{T_{\rm KT}}{T^*}\right), \tag{28}$$

where $\epsilon_F=(\pi/m)n$ is the Fermi energy of free fermions. It is useful to introduce reduced dimensionless temperatures $\tilde{T}_{\rm KT}\equiv T_{\rm KT}/\epsilon_F$ and $\tilde{T}^*=T^*/\epsilon_F$, which are small in the weak-coupling limit. Then we rewrite Eq. (28) as

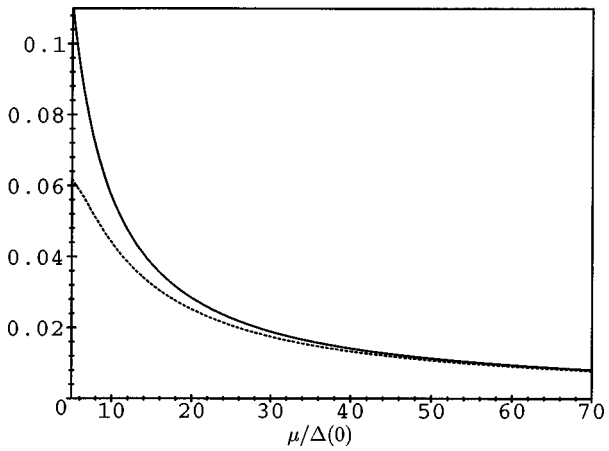

FIG. 1. Weak-coupling behavior of $T_{\rm KT}$. The solid line is T^*, and the dashed line represents $T_{\rm KT}$.

$$\tilde{T}_{\rm KT}\simeq\frac{1}{4}\frac{1}{1+1/4\tilde{T}^*}. \tag{29}$$

For small $\tilde{T}^*$ we may expand

$$\tilde{T}_{\rm KT}\approx\tilde{T}^*-4\tilde{T}^{*2}. \tag{30}$$

This equation shows explicitly how for decreasing coupling strength $T_{\rm KT}$ merges with T^*.

For weak coupling strengths, $T_{\rm KT}$ behaves like

$$\tilde{T}_{\rm KT}\approx\frac{e^\gamma}{\pi}\frac{\epsilon_F}{\Delta(0)}. \tag{31}$$

The merging of the two temperatures in the weak-coupling regime is displayed in Fig. 1.

Consider now the opposite limit of strong couplings. There we see from Eqs. (26), (2), (3), and (21) for $T_{\rm KT}$, $n(T,\mu)$, and $\Delta(T,\mu)$ that $T_{\rm KT}$ tends to a constant value. We can observe that in the strong-coupling limit $\Delta(T_{\rm KT})$ is always situated close to the zero-temperature value of $\Delta(T_{\rm KT},\mu)\approx\Delta(T=0,\mu)$. Taking this into account we derive an estimate for the second term in Eq. (21), thus obtaining the strong-coupling equation for $T_{\rm KT}$:

$$T_{\rm KT}\simeq\frac{\pi}{8}\left\{\frac{1}{m}n-\frac{T_{\rm KT}}{\pi}\exp\left[-\frac{\sqrt{\mu^2+\Delta^2(T_{\rm KT},\mu)}}{T_{\rm KT}}\right]\right\}. \tag{32}$$

With the approximation $\Delta(T_{\rm KT},\mu)\approx\Delta(T=0,\mu)$, we find that the first term in the exponent tends in the strong-coupling limit to a constant, $\Delta^2(T_{\rm KT},\mu)/2\mu T_{\rm KT}\to-4$, whereas the first term in the brackets tends to $-\infty$, so that Eq. (32) has the limiting form

$$T_{\rm KT}\simeq\frac{\pi}{8}\frac{n}{m}\left\{1-\frac{1}{8}\exp\left[\frac{2\mu}{\epsilon_F}-4\right]\right\}. \tag{33}$$

Thus for increasing coupling strength, the phase-decoherence temperature $T_{\rm KT}$ tends very quickly toward a constant:

$$T_{\rm KT}\simeq\frac{\pi}{8}\frac{n}{m}. \tag{34}$$

In this limit we know from Eq. (3) that the difference in the carrier density at zero temperature, $n(T=0)$, becomes equal to $n(T=T_{\rm KT})$, so that our limiting result coincides with that obtained in the ''fixed carrier density model'':

$$T_{\rm KT}=\frac{\epsilon_F(n_0)}{8}=\frac{\pi}{8m}n_0, \tag{35}$$

where we have inserted again $\epsilon_F(n)=(\pi/m)n$ for the Fermi energy of free fermions at the carrier density $n_0=n(T=0)$.

From the above asymptotic formulas for weak- and strong-coupling limits we see that the temperature of the Kosterlitz-Thouless transition is a monotonous function of coupling strength and carrier density. The crossover takes place in a narrow region $\mu/\Delta\in(-1,1)$. It is also observed in the behavior of the three-dimensional condensation temperature T_c of the gas of tightly bound, almost free, composite bosons in Refs. 2, 5, and 9. In the first two of these references, which include only quadratic fluctuations around the mean field (corresponding to ladder diagrams), T_c was shown to tend to a constant free Bose gas value $T_c=[n/2\zeta(3/2)]^{2/3}\pi/m$, with no dependence the internal structure of the boson. Here we find a similar result in two dimensions, where $T_{\rm KT}$ tends to a constant depending only on the mass $2m$ and the density $n/2$ of the pairs. No dependence on the coupling strength is left. The only difference with respect to the three-dimensional case is that here the transition temperature $T_c=T_{\rm KT}$ is linear in the carrier density n, while growing like $n^{2/3}$ in three dimensions. Our limiting result (35) agrees with Refs. 26 and 24. There exists a corresponding equation for the temperature T^* in the strong-coupling limit $\epsilon_0\gg\epsilon_F$:

$$T^*\simeq\frac{\epsilon_0}{2}\frac{1}{\ln\epsilon_0/\epsilon_F}. \tag{36}$$

IV. PHASE FLUCTUATIONS IN THREE DIMENSIONS

In this section we discuss, in a completely analogous way, the fluctuations in three dimensions, where the stiffness coefficient is, for small temperatures where $\Delta(T)$ is close to $\Delta(0)$,

$$J_{\rm 3D}(\mu,T,\Delta)=\frac{1}{4m}n(\mu,T,\Delta)-\frac{\sqrt{2m}}{16\pi^2}\frac{1}{T}\times\int_{-\mu}^{\infty}d\xi\frac{(\xi+\mu)^{3/2}}{\cosh^2(\sqrt{\xi^2+\Delta^2}/2T)}, \tag{37}$$

governing the phase fluctuations via an effective XY model

$$H=\frac{J_{\rm 3D}}{2}\int d^3\mathbf{x}[\nabla\theta(\mathbf{x})]^2. \tag{38}$$

The temperature of the phase transition in this model can reasonably be estimated using mean-field (MF) methods for the lattice three-dimensional (3D) XY model[8]

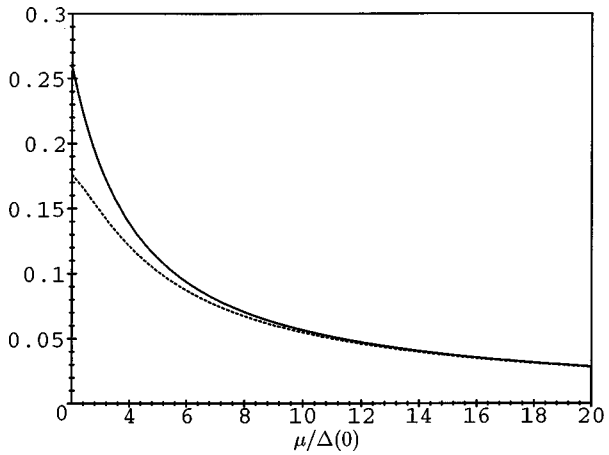

FIG. 2. Weak-coupling behavior of T_c in three dimensions. The solid line is T^*, T_c is plotted with dashed line.

$$T_{\rm 3D}^{\rm MF}\simeq 3J_{\rm 3D}a; \tag{39}$$

$a=1/n_b^{1/3}$ is the lattice spacing of the theory,[8] where n_b is number of pairs.

In the weak-coupling limit, the stiffness coefficient can be derived with the help of Gorkov's well-known method (setting $T_c\approx T^*$) as

$$J_{\rm 3D}=\frac{7}{48\pi^4}\zeta(3)\frac{p_F^3}{m}\frac{\Delta^2}{T^{*2}}. \tag{40}$$

This is precisely the coefficient of the gradient term in the Ginzburg-Landau expansion. In the weak-coupling limit, the two temperatures merge according to the formula

$$\tilde{T}_c=\tilde{T}^*-\alpha\tilde{T}^{*5/2}, \tag{41}$$

which contains a larger power of $\tilde{T}^*$ in the second term as well as a smaller prefactor $\alpha=(2\pi^2)^{2/3}/2\approx 3.65$, as compared with the two-dimensional separation formula (30). The merging behavior is displayed in Fig. 2.

In the strong-coupling limit of the theory where we have tightly bound composite bosons, the phase stiffness tends asymptotically to

$$J=\frac{n}{4m}-\frac{3\sqrt{2\pi m}}{16\pi^2}T^{3/2}\exp\left[-\frac{\sqrt{\mu^2+\Delta^2}}{T}\right]. \tag{42}$$

It obviously tends in this limit quickly to

$$J_{\rm BE}=\frac{n}{4m}. \tag{43}$$

An estimate for the critical temperature, obtained via the mean-field treatment of the 3D XY model on the lattice reads in this limit:

$$T_c=\frac{3}{2m}\left[\left(\frac{n}{2}\right)^{2/3}-\frac{1}{n^{1/3}}\frac{1}{2^{7/6}\pi^{3/2}}T_c^{3/2}m^{3/2}\exp\left(-\frac{\sqrt{\mu^2+\Delta^2}}{T_c}\right)\right]. \tag{44}$$

This quickly tends from below to the value

$$T_c^{3D\ XY}=\frac{3n^{2/3}}{2^{5/3}m}=\epsilon_F\frac{3}{(6\pi^2)^{2/3}}\simeq 0.2\epsilon_F. \tag{45}$$

This result is very close to the temperature of the condensation of bosons of mass $2m$ and density $n/2$, which, as was discussed in Sec. I, was obtained including the effect of Gaussian fluctuations into the mean-field equation for the particle number[2,5] yielding[44]

$$T_c^{\text{Bosons}}=[n/2\zeta(3/2)]^{2/3}\pi/m=0.218\epsilon_F. \tag{46}$$

V. CONCLUSION

We have studied the crossover from BCS to Bose-type superconductivity. For this purpose we have used the gradient expansion of the effective energy functional to set up an equivalent XY model which allows us to investigate the onset of long-range order in the phase fluctuations. In two dimensions, we have given a simple analytic expression which shows how the resulting Kosterlitz-Thouless temperature T_{KT} at which quasi-long-range order sets in moves towards the pair-binding temperature T^*, and merges with it in the weak-coupling limit. A similar expression was found in three dimensions. In the strong-coupling limit we find that the critical temperature tends in both two and three dimensions critical temperature to a constant value as the chemical potential changes it sign.

Let us finally remark that the separation of T^* and T_c has an analogy in the ferroelectrics and magnets which also contain two separate characteristic temperatures (for example, in the latter case, the Stoner and the Curie temperatures). It also can be studied more precisely in a simple field-theoretic model in $2+\epsilon$ dimensions with an $O(n)$ symmetry for large n. In such a model, the existence of two small parameters ϵ and $1/n$ has permitted us recently to *prove* the existence of two transitions, and to exhibit clearly their different physical origins.[45]

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