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Efficient algorithm for perturbative calculation of multi-loop Feynman integrals

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Abstract

We present an efficient algorithm for calculating multi-loop Feynman integrals perturbatively. © 2000 Elsevier Science B.V. All rights reserved.

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1. Recently, a new method has been proposed to calculate Feynman integrals of multi-loop diagrams perturbatively [1]. Together with the solution procedure for graphical recursion relations developed in Ref. [2], this should ultimately lead to the completely automatized computer generation of perturbation expansions of field theories up to high orders. Such expansions are needed in all strongly coupled fluctuating field systems, for example those describing the critical phenomena close to second-order phase transitions (see Ref. [3] and Addendum [4]). So far, expansions have been limited to seven loops only [5,6], which are barely sufficient to yield critical exponents [7] with an accuracy comparable to experimental data [8].

In this Letter, we would like to show how the expansions proposed in Ref. [1] can be performed most efficiently, such that they can be carried out on a computer to high orders in a limited computer time.

2. A basic Feynman integral with L loops, n internal lines, and E external momenta $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_E$ has the form

$$I^D_{a_k} = \int \frac{\mathrm{d}^D p_1}{(2\pi)^D} \cdots \frac{\mathrm{d}^D p_L}{(2\pi)^D} \prod_{k=1}^{n} \frac{1}{\left(1+\boldsymbol{q}_k^2\right)^{a_k}}, \quad n \geq L, \tag{1}$$

with some powers a_k, where $\boldsymbol{q}_k$ are the momenta carried by the lines, and the integrations run over all loop momenta $\boldsymbol{p}_i$. The line momenta $\boldsymbol{q}_k$ are linear combinations of the loop momenta $\boldsymbol{p}_i$ and the external momenta $\boldsymbol{k}_j$.

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For simplicity, we have set all masses equal to unity. A Feynman integral with different non-zero masses can be reduced to (1) by an appropriate rescaling of the line momenta $\boldsymbol{q}_k$. Following Ref. [1], we view the integral (1) as a special case $I^D_{a_k} = I^D_{a_k}(1)$ of the function

$$I^D_{a_k}(\kappa) = \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_L}{(2\pi)^D} \prod_{k=1}^{n} \frac{e^{a_k(\kappa-1)\boldsymbol{q}_k^2}}{\left(1+\kappa \boldsymbol{q}_k^2\right)^{a_k}} \tag{2}$$

to be calculated perturbatively via a Taylor series expansions in powers of κ.

It is the purpose of this Letter to point out that the simplest way to derive such an expansion is by rewriting each generalized propagator in the Schwinger parametric form

$$\frac{e^{a(\kappa-1)\boldsymbol{q}^2}}{\left(1+\kappa \boldsymbol{q}^2\right)^{a}} = \frac{1}{\Gamma(a)} \int_0^\infty dt\, t^{a-1} e^{-t} e^{[a(\kappa-1)-\kappa t]\boldsymbol{q}^2}. \tag{3}$$

Then the integrals (2) take the form

$$I^D_{a_k}(\kappa) = \frac{1}{\Gamma(a_1)} \cdots \frac{1}{\Gamma(a_n)} \int_0^\infty dt_1\, t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_n\, t_n^{a_n-1} e^{-t_n} \int \frac{d^D p_1}{(2\pi)^D} \cdots \int \frac{d^D p_L}{(2\pi)^D}$$
$$\times \exp\left\{ -\sum_{k=1}^{n} \left[a_k(1-\kappa) + \kappa t_k \right] \boldsymbol{q}_k^2 \right\}. \tag{4}$$

Collecting the L loop momenta $\boldsymbol{p}_i$ and the E external momenta $\boldsymbol{k}_j$ in single vector symbols

$$p = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_L), \quad k = (\boldsymbol{k}_1, \ldots, \boldsymbol{k}_E), \tag{5}$$

we rewrite

$$\sum_{k=1}^{n} \left[a_k(1-\kappa) + \kappa t_k \right] \boldsymbol{q}_k^2 = \tfrac{1}{2} p^T M p + p^T M' k + \tfrac{1}{2} k^T M'' k \tag{6}$$

and complete the squares to

$$\sum_{k=1}^{n} \left[a_k(1-\kappa) + \kappa t_k \right] \boldsymbol{q}_k^2 = \tfrac{1}{2}\left(p + M^{-1}M'k \right)^T M \left(p + M^{-1}M'k \right) + \tfrac{1}{2} k^T \left(M'' - M'^T M^{-1} M' \right) k, \tag{7}$$

with symmetric matrices M and M''. After a shift $p \to p - M^{-1}M'k$ of integration variables, the $\boldsymbol{p}_k$ integrations become Gaussian, and we obtain

$$I^D_{a_k}(\kappa) = \frac{1}{\Gamma(a_1) \cdots \Gamma(a_n)} \int_0^\infty dt_1\, t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_n\, t_n^{a_n-1} e^{-t_n} \exp\left(-\tfrac{1}{2} k^T \left(M'' - M'^T M^{-1} M' \right) k \right)$$
$$\times \int \frac{d^D p_1}{(2\pi)^D} \cdots \int \frac{d^D p_L}{(2\pi)^D} e^{-\frac{1}{2} p^T M p}$$
$$= \frac{(2\pi)^{-LD/2}}{\Gamma(a_1) \cdots \Gamma(a_n)} \int_0^\infty dt_1\, t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_n\, t_n^{a_n-1} e^{-t_n} \frac{\exp\left(-\tfrac{1}{2} k^T \left(M'' - M'^T M^{-1} M' \right) k \right)}{(\det M)^{D/2}}, \tag{8}$$

where the matrices M, M', and M'' depend on κ and the t_k through linear combinations

$$c_k(\kappa,t_k) \equiv a_k(1-\kappa) + \kappa t_k. \tag{9}$$

Although the entries of the matrix M depend on the routing of the loop momenta through the different lines, the determinant of M is invariant under changes of the routing, except for trivial relabelings of the a_k.

In order to derive the desired expansion of $I^D_{a_k}(\kappa)$ in powers of κ, we expand the integrand on the right hand side of (8) in powers of κ, whose coefficients are polynomials in the parameters t_i $(i=1,\dots,L)$. The t_i-integrals can then all be performed using the formula

$$\int_0^\infty \mathrm{d}t\, t^\gamma \mathrm{e}^{-t} = \Gamma(\gamma+1). \tag{10}$$

For diagrams without external momenta, appearing in the perturbation expansions for the ground state of quantum field theories, (8) simplifies to

$$I^D_{a_k}(\kappa) = \frac{(2\pi)^{-LD/2}}{\Gamma(a_1)\cdots\Gamma(a_n)} \int_0^\infty \mathrm{d}t_1\, t_1^{a_1-1}\mathrm{e}^{-t_1} \cdots \int_0^\infty \mathrm{d}t_n\, t_n^{a_n-1}\mathrm{e}^{-t_n} (\det M)^{-D/2}. \tag{11}$$

More general Feynman integrals than those in Eq. (1) may contain loop momenta $\boldsymbol{p}_k$ in the numerator of the integrand. These can be calculated with a simple extension of the above technique, by introducing ''source terms'' $\sum_{i=1}^{L} \boldsymbol{j}_i \cdot \boldsymbol{p}_i$ into the exponents of (2) and (3), and appropriately differentiate the resulting κ-expansion with respect to $\boldsymbol{j}_i$, which are set equal to zero at the end.

3. As a first example, take the exactly solvable one-loop integral

$$I^D_a = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{1}{(1+\boldsymbol{p}^2)^a} = \frac{\Gamma(a-D/2)}{(4\pi)^{D/2}\Gamma(a)}. \tag{12}$$

Its κ-generalized version can be expressed in terms of a confluent hypergeometric function,

$$\begin{aligned} I^D_a(\kappa) &\equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{\mathrm{e}^{a(\kappa-1)\boldsymbol{p}^2}}{(1+\kappa\boldsymbol{p}^2)^a} = \frac{1}{(4\pi\kappa)^{D/2}}\, \Psi\!\left(\frac{D}{2}, 1+\frac{D}{2}-a; \frac{a(1-\kappa)}{\kappa}\right) \\ &= \frac{1}{(4\pi\kappa)^{D/2}} \left[\frac{\Gamma(a-D/2)}{\Gamma(a)}\, {}_1F_1\!\left(\frac{D}{2}, 1+\frac{D}{2}-a; \frac{a(1-\kappa)}{\kappa}\right) \right. \\ &\quad \left. + \frac{\Gamma(D/2-a)}{\Gamma(D/2)} (1-\kappa)^{a-D/2}\, {}_1F_1\!\left(a, 1+a-\frac{D}{2}; \frac{a(1-\kappa)}{\kappa}\right) \right] \end{aligned} \tag{13}$$

with

$${}_1F_1(\alpha;\beta;z) \equiv \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{z^k}{k!}, \quad (a)_s \equiv \frac{\Gamma(a+s)}{\Gamma(a)} = \prod_{r=0}^{s-1} (\alpha+r) \quad (\text{Pochhammer's symbol}). \tag{14}$$

In Ref. [1] this was calculated perturbatively via a Wick expansion. Here we use our general formula (11) for vacuum integrals. The number of loops is $L=1$, and we identify

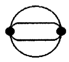
$$q_1 = p, \quad a_1 = a, \quad c_1 = a(1-\kappa) + \kappa t, \quad M = 2(c_1), \quad \det M = 2c_1. \tag{15}$$

Expanding $(\det M)^{-D/2}$ in powers of κ, and performing the resulting integrals over t in Eq. (11), we find directly the perturbation expansion for the loop integrals (12) in any dimension D:

$$\begin{aligned} I_a^D(\kappa) &= \frac{1}{(4\pi a)^{D/2}}\left[1+\frac{D(2+D)\kappa^2}{8a}-\frac{D(2+D)(4+D)\kappa^3}{24a^2}+\frac{(2+a)D(2+D)(4+D)(6+D)\kappa^4}{128a^3}\right. \\ &\quad \left.-\frac{(6+5a)D(2+D)(4+D)(6+D)(8+D)\kappa^5}{960a^4}+\mathscr{O}(\kappa^6)\right]. \end{aligned} \tag{16}$$

The expansion can easily extended any desired order. It agrees, of course, with what we would obtain from the exact expression (13) via a large-argument expansion of the confluent hypergeometric function.

4. As a nontrivial example, take the integral of the watermelon diagram treated in Ref. [1] only in $D=2$ dimensions:

$$I^D = $$  $$= \int \frac{d^Dp_1}{(2\pi)^D}\frac{d^Dp_2}{(2\pi)^D}\frac{d^Dp_3}{(2\pi)^D}\frac{1}{1+\mathbf{p}_1^2}\frac{1}{1+\mathbf{p}_2^2}\frac{1}{1+\mathbf{p}_3^2}\frac{1}{1+(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)^2}. \tag{17}$$

This integral has the powers

$$a_1=a_2=a_3=a_4=1, \tag{18}$$

and we identify the line momenta as

$$q_1=p_1,\quad q_2=p_2,\quad q_3=p_3,\quad q_4=p_1+p_2+p_3, \tag{19}$$

such that the matrix M is

$$M=2\begin{pmatrix} a_1+a_4 & a_4 & a_4 \\ a_4 & a_2+a_4 & a_4 \\ a_4 & a_4 & a_3+a_4 \end{pmatrix}, \tag{20}$$

$$\det M = 8(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4). \tag{21}$$

For the function $I_{a_k}^D(\kappa)$, we then obtain in any dimension D

$$\begin{aligned} I^D(\kappa) &= \frac{1}{2^{4D}\pi^{3D/2}}\left[1+\frac{9D(2+D)\kappa^2}{32}-\frac{9D(2+D)(4+D)\kappa^3}{128}+\frac{3D(2+D)(1048+522D+81D^2)\kappa^4}{4096}\right. \\ &\quad -\frac{9D(2+D)(4+D)(2576+918D+117D^2)\kappa^5}{40960}+\frac{D(2+D)(564864+397744D+110916D^2+15228D^3+891D^4)\kappa^6}{65536} \\ &\quad -\frac{3D(2+D)(4+D)(29651840+15696528D+3452148D^2+391068D^3+19683D^4)\kappa^7}{9175040} \\ &\quad \left.+\frac{3D(2+D)(4+D)(1419854080+843338336D+212508840D^2+29562300D^3+2344950D^4+85779D^5)\kappa^8}{83886080}+\mathscr{O}(\kappa^9)\right]. \end{aligned} \tag{22}$$

For $D=1$ this reduces to

$$I^1(\kappa)=\frac{1}{2^4\pi^{3/2}}\left[1+\frac{27\kappa^2}{32}-\frac{135\kappa^3}{128}+\frac{14859\kappa^4}{4096}-\frac{97497\kappa^5}{8192}+\frac{3268929\kappa^6}{65536}-\frac{63271629\kappa^7}{262144}\right.$$
$$\left.+\frac{22569248565\kappa^8}{16777216}+\mathscr{O}(\kappa^9)\right], \tag{23}$$

and for $D=2$ to

$$I^2(\kappa)=\frac{1}{2^8\pi^3}\left[1+\frac{9\kappa^2}{4}-\frac{27\kappa^3}{8}+\frac{453\kappa^4}{32}-\frac{1647\kappa^5}{32}+\frac{15157\kappa^6}{64}-\frac{157293\kappa^7}{128}\right.$$
$$\left.+\frac{3720699\kappa^8}{512}+\mathscr{O}(\kappa^9)\right], \tag{24}$$

thus extending easily the expansions in Ref. [1].

For $D=3$, the expansion reads

$$I^3(\kappa)=\frac{1}{2^{12}\pi^{9/2}}\left[1+\frac{135\kappa^2}{32}-\frac{945\kappa^3}{128}+\frac{150435\kappa^4}{4096}-\frac{1206387\kappa^5}{8192}+\frac{48595005\kappa^6}{65536}-\frac{1079675235\kappa^7}{262144}\right.$$
$$\left.+\frac{432899207685\kappa^8}{16777216}+\mathscr{O}(\kappa^9)\right]. \tag{25}$$

5. Having developed the tools for finding perturbation expansions of Feynman integrals, it remains to study the large-order behavior, and to find suitable methods for the resummation of the expansions with high accuracy. Together with the automatized generation of the Feynman diagrams of Ref. [2], this will open the way for an 'industrial production' of high-loop expansions for critical exponents,

References

[1] H. Kleinert, Perturbative Calculation of Multi-Loop Feynman Diagrams. New Type of Expansions for Critical Exponents, Berlin Preprint 1999, hep-th/9908078.
[2] H. Kleinert, A. Pelster, B. Kastening, M. Bachmann, Recursive Graphical Construction of Feynman Diagrams and Their Multiplicities in ϕ^4- and ϕ^2A-Theory, Berlin Preprint 1999, hep-th/9907168.
[3] H. Kleinert, Phys. Rev. D 57 (1998) 2264, e-print aps1997jun25_001.
[4] H. Kleinert, Phys. Rev. D 58 (1998) 107702, cond-mat/9803268.
[5] D.B. Murray, B.G. Nickel, unpublished.
[6] H. Kleinert, Phys. Rev. D 60 (1999) 085001, hep-th/9812197.
[7] H. Kleinert Theory and Satellite Experiment on Critical Exponent α of Specific Heat in Superfluid Helium, FU-Berlin preprint 1999, cond-mat/9906107.
[8] J.A. Lipa, D.R. Swanson, J. Nissen, T.C.P. Chui, U.E. Israelson, Phys. Rev. Lett. 76 (1996) 944.