Cosmology with Curvature-Saturated Gravitational Lagrangian $R/\sqrt{1 + l^4 R^2}$

Hagen Kleinert[1] and Hans-Jürgen Schmidt[2]

We argue that the Lagrangian for gravity should remain bounded at large curvature, and interpolate between the weak-field tested Einstein-Hilbert Lagrangian $\mathcal{L}_{EH} = R/16\pi G$ and a pure cosmological constant for large R with the curvature-saturated ansatz $\mathcal{L}_{cs} = \mathcal{L}_{EH}/\sqrt{1 + l^4 R^2}$, where l is a length parameter expected to be a few orders of magnitude above the Planck length. The curvature-dependent effective gravitational constant defined by $d\mathcal{L}/dR = 1/16\pi G_{eff}$ is $G_{eff} = G\sqrt{1 + l^4 R^2}^3$, and tends to infinity for large R, in contrast to most other approaches where $G_{eff} \to 0$. The theory possesses neither ghosts nor tachyons, but it fails to be linearization stable. In a curvature saturated cosmology, the coordinates with $ds^2 = a^2[da^2/B(a) - dx^2 - dy^2 - dz^2]$ are most convenient since the curvature scalar becomes a linear function of $B(a)$. Cosmological solutions with a singularity of type $R \to \pm\infty$ are possible which have a bounded energy-momentum tensor everywhere; such a behaviour is excluded in Einstein's theory. In synchronized time, the metric is given by

$$ds^2 = dt^2 - t^{6/5}(dx^2 + dy^2 + dz^2).$$

On the technical side we show that two different conformal transformations make $\mathcal{L}_{cs}$ asymptotically equivalent to the Gurovich-ansatz $\mathcal{L} = |R|^{4/3}$ on the one hand, and to Einstein's theory with a minimally coupled scalar field with self-interaction on the other.

KEY WORDS: Cosmology; effective gravitational constant; fourth-order gravity.

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1. INTRODUCTION

According to an old idea by Sakharov [1], the gravitational properties of spacetime are caused by the bending stiffness of all quantum fields in a spacetime of scalar curvature R. This idea of induced gravity has inspired many subsequent theories of gravitation, from Adler's [2] proposal to consider Einstein gravity as a symmetry breaking effect in quantum field theory to the modern induced gravity derived from string fluctuations [3]. Whatever the precise mechanism, any induced gravity will lead to a Lagrangian which is bounded at large R, and may also go to zero. The latter case would be analogous to the elastic stiffness of solids, which is constant for small distortions, but vanishes after the solid cracks.

In this paper we investigate the physical consequences of a simple Lagrangian which goes to a constant at large R, thus interpolating between the Einstein-Hilbert Lagrangian for small R and a pure cosmological constant for large R. This Lagrangian will be referred to as *curvature-saturated* and reads

$$\mathcal{L}_{cs} = \frac{1}{16\pi G} \frac{R}{\sqrt{1 + l^4 R^2}}. \tag{1.1}$$

The length parameter l may range from an order of the Planck length l_P or a few orders of magnitude larger than l_P. Applying standard methods and those of Refs. [4–8], we shall derive the cosmological consequences of the saturation and compare our ansatz with others.

One of the motivations for a renewed interest in a more detailed consideration of cosmology with non-linear curvature terms comes from M-theory, see Ref. [9] "Brane new world." In [9] a conformal anomaly is considered, which turns out to have analogous consequences as Starobinsky's anomaly-driven inflation with R– and R^2-terms, see e.g. Refs. [10] for the older results. Ref. [11] contains the latest results concerning the effective Λ-term in such models.

Our own direct motivation to tackle the model discussed below was as follows: We tried to make the analogy proposed in [1] more closer than done by others; the analogy with solid state physics is this one: For small forces, the resistance to bending is proportional to this force, but after a certain threshold – defined by cracking the solid – the resistance vanishes.

A similar line of reasoning was deduced in Ref. [12]: There the finite-size effects from the closed Friedmann universe to the quantum states of fields have been calculated. Instead of continuous distribution of the energy levels of the quantum fields, one has a discrete spectrum. Qualitatively, the result is: If the radius a of the spatial part of spacetime shrinks close to zero, which is almost the same as very large R, then the spacings between the energy levels become larger and larger, and after a certain threshold, all fields will be in the ground state. This behaviour shall be represented by an effective action. The concrete form of the corresponding effective Lagrangian is not yet fully determined (that shall be the topic of later

work), but preliminarily we found out that the behaviour for large R will quite probably be of a Lagrangian bounded by a special effective Λ; so we have chosen one of the easiest analytic functions possessing this large-R behaviour together with the correct weak-field shape.

The paper is organized as follows: In Sec. 2 we calculate the consequences of the effective Lagrangian $\mathcal{L}_{cs}$.

In Sec. 3 we investigate the consequences of the R-dependence of the effective gravitational constant defined by

$$\frac{1}{16\pi G_{eff}} \equiv \frac{d\mathcal{L}}{dR}, \tag{1.2}$$

which is

$$G_{eff} = G\sqrt{1+l^4 R^2}^{\,3} \tag{1.3}$$

for $\mathcal{L} = \mathcal{L}_{cs}$ and tends to infinity as $R \to \pm\infty$.

Then we apply two different conformal transformations to $\mathcal{L}_{cs}$. One of them, presented in Sec. 4, makes $\mathcal{L}_{cs}$ asymptotically equivalent to the Gurovich-ansatz [13], [14]

$$\mathcal{L} = \frac{R}{16\pi G} + c_1 |R|^{4/3}. \tag{1.4}$$

The other transformation, by the Bicknell theorem given in Sec. 5, establishes a conformal relation to Einstein's theory, with a minimally coupled scalar field. In the literature, see [15] and the references cited there, only the second of these conformal transformations has so far been used. The physical consequences of these three theories are, of course, quite different since the metrics are not related to each other by coordinate transformations.

Our approach differs fundamentally from that derived from the *limiting curvature hypothesis* (LCH) in Refs. [16], where the gravitational Lagrangian reads

$$\mathcal{L} = R + \frac{\Lambda}{2}\left(\sqrt{1 - R^2/\Lambda^2} - 1\right) \tag{1.5}$$

whose derivative with respect to R diverges for $R \to \Lambda$. This divergence was supposed to prevent a curvature singularity, a purpose not completely reached by the model presented in the first of Refs. [16] because other curvature invariants may still diverge. (Let us note for completeness: In the second of Refs. [16], a more detailed version of the LCH is presented which covers also the bounding of the other curvature invariants; it is restricted to isotropic cosmological models. For more general space-times one faces the problem that sometimes a curvature singularity exists, but all polynomial curvature invariants remain bounded there.)

In contrast to Eq. (1.5), our model *favors* high curvature values.

It turns out that the use of synchronized or conformal time is not optimal for our problem. We therefore use a new time coordinate which we call *curvature time* for the spatially flat Friedmann model. The general properties of this coordinate choice are described in Sec. 6.

In Sec. 7 we study the consequences of curvature-saturation for some cosmological models using the coordinates of Sec. 6. In Sec. 8, finally, we summarize our results and compare with the related papers [17] to [31].

2. FIELD EQUATIONS OF CURVATURE-SATURATED GRAVITY

The curvature-saturated Lagrangian (1.1) interpolates between the Einstein-Hilbert Lagrangian

$$\mathcal{L}_{\text{EH}} = \frac{R}{16\pi G}, \tag{2.1}$$

which is experimentally confirmed at weak fields, and a pure cosmological constant at strong fields

$$\mathcal{L} = \frac{\pm 1}{16\pi G l^2}. \tag{2.2}$$

The R dependence is plotted in Fig. 1.

The usual gravitational constant is obtained from the derivative of the Einstein-Hilbert Lagrangian:

$$\frac{1}{16\pi G} = \frac{d\mathcal{L}_{\text{EH}}}{dR}. \tag{2.3}$$
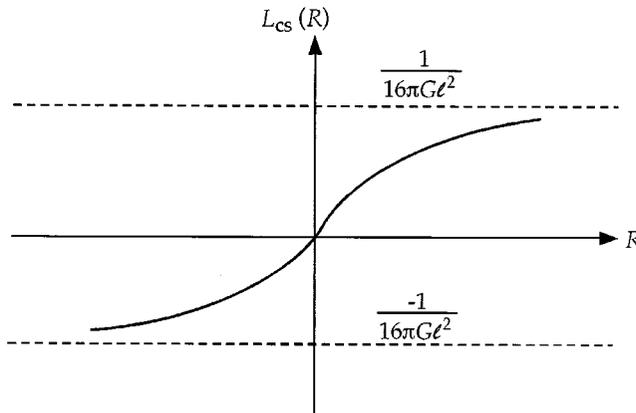

Figure 1. Curvature-Saturated Lagrangian $\mathcal{L}_{\text{cs}}$ as a function of the curvature scalar R.

From our curvature-saturated Lagrangian (1.1) we obtain, with this derivative, the effective gravitational constant (1.3). The definition (2.3) is motivated as follows: If one considers the Newtonian limit for a general Lagrangian $\mathcal{L}(R)$ which may contain a nonvanishing cosmological constant, the potential between two point masses contains a Newtonian $1/r$-part plus a Yukawa-like part $\exp(-r/r_Y)$ stemming from the nonlinearities of the Lagrangian; the details are given in Appendix A. At distances much larger than r_Y, but much smaller than $1/\sqrt{R}$, only the $1/r$-term survives, and the coupling strength of the $1/r$-term is given by the effective gravitational constant G_{eff}. For a recent version to deduce such weak-field expressions, see Ref. [24].

For a general Lagrangian $\mathcal{L}(R)$ such as (1.1), the calculation of the field equation is somewhat tedious, since the Palatini formalism which simplifies the calculation in Einstein's theory is no longer applicable. Recall that in this, metric and the affine connection are varied independently, the latter being identified with the Christoffel symbol only at the end.

Here the following indirect procedure leads rather efficiently to the correct field equations. Let

$$\mathcal{L}' \equiv \frac{d\mathcal{L}}{dR}, \quad \mathcal{L}'' \equiv \frac{d^2\mathcal{L}}{dR^2}, \tag{2.4}$$

and form the covariant energy-momentum tensor of the gravitational field which is given by the variational derivative of $\mathcal{L}$ with respect to the metric g_{ab}:

$$\Theta^{ab} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}\sqrt{-g}}{\delta g_{ab}}, \tag{2.5}$$

where g denotes the determinant of g_{ab}. For dimensional reasons, Θ_{ab} has the following structure

$$\Theta_{ab} = \alpha \mathcal{L}' R_{ab} + \beta \mathcal{L}' R g_{ab} + \gamma \mathcal{L} g_{ab} + \delta \Box \mathcal{L}' g_{ab} + \epsilon \mathcal{L}'_{;ab} \tag{2.6}$$

with the 5 real constants $\alpha \dots \epsilon$. These constants can be uniquely determined up to one overall constant factor by the covariant conservation law

$$\Theta^{ab}_{;b} = 0. \tag{2.7}$$

The overall factor is fixed by the Einstein limit $l \to 0$ of the theory, where $\Theta_{ab} = (R_{ab} - \frac{1}{2}Rg_{ab})/8\pi G$. In this way we derive the following form of the covariantly conserved energy-momentum tensor of the gravitational field

$$\Theta_{ab} = \frac{1}{16\pi G}(2\mathcal{L}' R_{ab} - \mathcal{L} g_{ab} + 2\Box\mathcal{L}' g_{ab} - 2\mathcal{L}'_{;ab}). \tag{2.8}$$

The calculation is straightforward, if one is careful to distinguish between $(\Box \mathcal{L}')_{;a}$ and $\Box(\mathcal{L}'_{;a})$, which differ by a multiple of the curvature scalar.

Inserting our curvature-saturated Lagrangian (1.1) into (2.4) and omitting the subscript, we have

$$\mathcal{L} = \frac{R}{2}(1 + l^4 R^2)^{-1/2}, \quad \mathcal{L}' = \frac{d\mathcal{L}}{dR} = \frac{1}{2}(1 + l^4 R^2)^{-3/2}, \tag{2.9}$$

and find from (2.8)

$$\Theta_{ab} = \frac{1}{8\pi G}\left\{ \frac{R_{ab}}{(1+l^4R^2)^{3/2}} - \frac{Rg_{ab}}{2(1+l^4R^2)^{1/2}} + g_{ab}\square\left[\frac{1}{(1+l^4R^2)^{3/2}}\right] \right.$$
$$\left. - \left[\frac{1}{(1+l^4R^2)^{3/2}}\right]_{;ab} \right\}. \tag{2.10}$$

Setting $l = 0$ reduces this to $1/16\pi G$ times the Einstein tensor. The trace of (2.10) is

$$\Theta_a{}^a = \frac{1}{8\pi G}\left\{ \frac{R + 2l^4 R^3}{(1+l^4R^2)^{3/2}} - 3\square\left[\frac{1}{(1+l^4R^2)^{3/2}}\right] \right\}. \tag{2.11}$$

According to Einstein's equation, Θ_{ab} has to be equal to the energy momentum tensor of the matter T_{ab}, i.e., $T_{ab} = \Theta_{ab}$. Equation (2.11) implies that in the vacuum, the only constant curvature scalar is $R = 0$, such that this model does not possess a de Sitter solution. Further, we can see from Eq. (2.10), that a curvature singularity does not necessarily imply a divergence of energy-momentum, but may be compensated by the infinity of G_{eff}.

3. EFFECTIVE GRAVITATIONAL CONSTANT AND WEAK-FIELD BEHAVIOR

Let us compare the effective gravitational constant G_{eff} of our curvature-saturated model with those of other models discussed in the literature. From (1.3) we see that G_{eff} has the weak-field expansion

$$G_{\text{eff}} = G\left(1 + \frac{3}{2}l^4 R^2 + \cdots\right), \tag{3.1}$$

and the strong-field expansion

$$G_{\text{eff}} = Gl^6|R|^3\left(1 + \frac{3}{2l^4 R^2} + \cdots\right). \tag{3.2}$$

The full R-behavior is plotted in Fig. 2.

The weak-field expansion of $\mathcal{L}_{\text{cs}}$ is given by

$$\mathcal{L}_{\text{cs}} = \frac{R}{16\pi G\sqrt{1+l^4R^2}} = \frac{R}{16\pi G} + \sum_{k=1}^{\infty} b_k\, R^{2k+1} \tag{3.3}$$

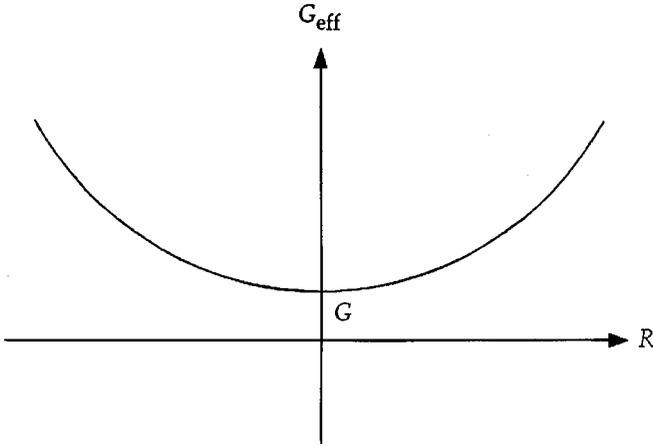with real coefficients b_k, where $b_1 = -l^4/32\pi G$.

Figure 2. Effective gravitational constant as a function of the curvature scalar.

As one can see, the quadratic term is absent, so that the linearized field equation coincides with the linearized Einstein equation. Thus we encounter neither ghosts nor tachyons; for details see Appendix B.

There is, however, a price to pay for it. The theory has lost linearization stability of the solutions. This latter property has the following consequences: If one performs a weak-field expansion

$$g_{ij} = \eta_{ij} + \sum_{m=1}^{\infty} \epsilon^m g_{ij}^{(m)} \tag{3.4}$$

around flat spacetime to solve the field equation, one has to use the terms up to the order $m = 2$ to get the complete weak-field part of the set of solutions. With this peculiarity, we obtain a well-posed Cauchy problem for the gravity theory following from the Lagrangian $\mathcal{L}_{cs}$.

Let us now compare our theory with others available in the literature. Let

$$\mathcal{L}_{\alpha,n}(R) = \frac{R}{16\pi G} + \alpha R^n \tag{3.5}$$

with some number $n > 1$ and constant $\alpha \neq 0$. In analogy with Eq. (1.2) we calculate the effective gravitational constant from

$$\frac{1}{16\pi G_{\text{eff}}} = \frac{d\mathcal{L}_{\alpha,n}}{dR} = \frac{1}{16\pi G} + \alpha n R^{n-1} \tag{3.6}$$

such that

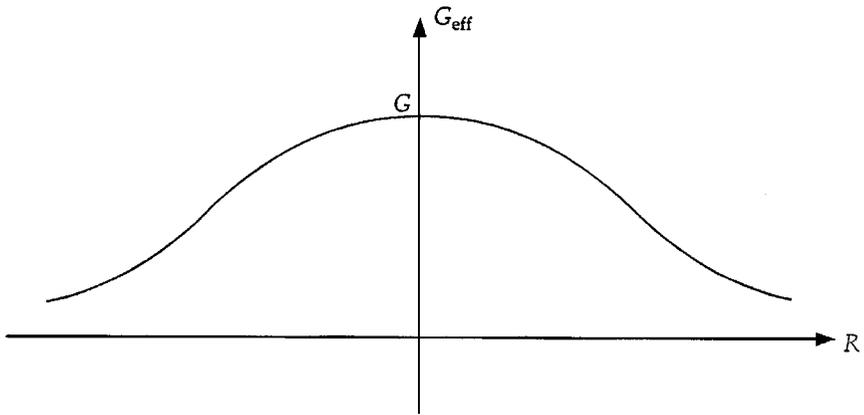$$G_{\text{eff}} = \frac{G}{1 + 16\pi \alpha n R^{n-1} G}, \tag{3.7}$$

Figure 3. Effective gravitational constant G_{eff} for $\mathcal{L}_{\alpha,3}$ with $\alpha > 0$ as a function of R.

i.e., $G_{\text{eff}} \to 0$ as $R \to \pm\infty$. For $n = 2$, more exactly: for all even natural numbers n, we meet an additional peculiarity that G_{eff} can diverge for finite values of R already. Such values of $R = R_{\text{crit}}$ are called *critical* [4]. For $n = 2$ we get

$$R_{\text{crit}} = -\frac{1}{32\alpha\pi G}, \tag{3.8}$$

and this is the region where G_{eff} changes its sign, as shown in Figures 3 and 4. At critical values of the curvature scalar, the Cauchy problem fails to be a well-posed one.

4. CONFORMAL DUALITY

In Ref. [8], a duality transformation relating between different types of non-linear Lagrangians has been found. In the present notation it implies the following relation. Let

$$\hat{g}_{ab} = \mathcal{L}'^2 g_{ab} \tag{4.1}$$

be the conformally transformed metric with $\mathcal{L}' \neq 0$, which is fulfilled by our Lagrangian (1.1). Then the conformally transformed curvature scalar equals

$$\hat{R} = \frac{3R}{\mathcal{L}'^2} - \frac{4\mathcal{L}}{\mathcal{L}'^3}, \tag{4.2}$$

and the associated Lagrangian is

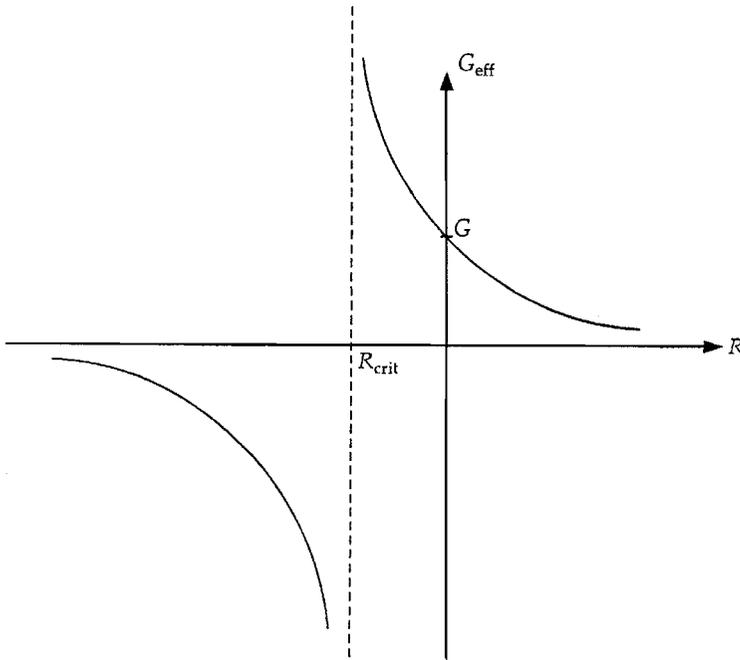$$\hat{\mathcal{L}} = \frac{2R}{\mathcal{L}'^3} - \frac{3\mathcal{L}}{\mathcal{L}'^4}. \tag{4.3}$$

Figure 4. Effective gravitational constant G_{eff} for $\mathcal{L}_{\alpha,2}$ with $\alpha > 0$ as a function of R.

We easily verify that $\hat{\mathcal{L}}'\,\mathcal{L}' = 1$. Then one can prove that g_{ab} solves the vacuum field equation following from $\mathcal{L}(R)$ if and only if $\hat{g}_{ab}$ of Eq. (4.1) solves the corresponding equation for $\hat{\mathcal{L}}(\hat{R})$ of Eq. (4.3).

Example: For $\mathcal{L} = R^{k+1}$ we find, up to an inessential constant factor, $\hat{\mathcal{L}} = \hat{R}^{\hat{k}+1}$ with $\hat{k} = 1/(2 - 1/k)$, such that for a purely quadratic theory with $\mathcal{L} = R^2$, also $\hat{\mathcal{L}} = \hat{R}^2$. For our curvature-saturated model $\mathcal{L} \to$ const. we should expect a behavior with $k \to -1$, i.e., $\hat{k} \to 1/3$, this leads to $\hat{\mathcal{L}} \sim \hat{R}^{4/3}$, which is the Gurovich-model [13], cf. Eq. (1.4).

Let us study this in more detail. To simplify the expressions we use, in this subsection only, reduced units with $16\pi\,G = 1$ to best exhibit the fixed point $l = 0$ of this transformation making it an identity transformation if applied to Einstein's theory where $k = 1$. In the present units, Eqs. (2.9) have to be multiplied by 2 and become

$$\mathcal{L} = R(1 + l^4R^2)^{-1/2}, \quad \mathcal{L}' = \frac{d\mathcal{L}}{dR} = (1 + l^4R^2)^{-3/2}. \tag{4.4}$$

Inserting these into (4.1)–(4.3), we obtain

$$\hat{g}_{ab} = \frac{g_{ab}}{(1 + l^4R^2)^3} \tag{4.5}$$

and

$$\hat{R} = -R(1 + l^4 R^2)^3(1 - 4l^4 R^2). \tag{4.6}$$

For small R we have

$$\hat{R} = -R(1 - l^4 R^2 + \cdots), \tag{4.7}$$

and for large $|R|$

$$\hat{R} = 4l^{16} R^9 \left(1 + \frac{11}{4l^4 R^2} + \cdots\right). \tag{4.8}$$

The inverse function $R(\hat{R})$ of (4.6) is not expressible in closed form, but its small- and large-curvature expansion can be calculated from (4.7) and (4.8)

$$R = -\hat{R}(1 + l^4 \hat{R}^2 + \cdots), \quad R = \left(\frac{\hat{R}}{4l^{16}}\right)^{1/9} \left[1 - \frac{11}{36l^4}\left(\frac{4l^{16}}{\hat{R}}\right)^{2/9} + \cdots\right] \tag{4.9}$$

From Eq. (4.3) we see that

$$\hat{\mathcal{L}} = -R(1 + l^4 R^2)^{9/2}(1 - 3l^4 R^2) \tag{4.10}$$

where $R(\hat{R})$ has to be inserted. For large R we use the right-hand equation in (4.9) and obtain the limiting behavior

$$\hat{\mathcal{L}} = 3l^{22}\left(\frac{\hat{R}}{4l^{16}}\right)^{4/3} \left[1 - \frac{51}{6l^4}\left(\frac{4l^{16}}{\hat{R}}\right)^{2/9} + \cdots\right]. \tag{4.11}$$

5. BICKNELL'S THEOREM

Bicknell's theorem [25], in the form described in Ref. [4], relates Lagrangians of the type (2.9) to Einstein's theory coupled minimally to a scalar field ϕ with a certain interaction potential $\tilde{V}(\phi)$. This Lagrangian is given by

$$\mathcal{L}_{EH} + \frac{1}{2}\phi_{,i}\phi^{,i} - \tilde{V}(\phi). \tag{5.1}$$

The relation of $\tilde{V}(\phi)$ with $\mathcal{L}(R)$ is expressed most simply by defining a field with a different normalization $\psi = \sqrt{2/3}\,\phi$, in terms of which the potential $\tilde{V}(\phi) = V(\psi)$ reads

$$V(\psi) = \mathcal{L}(R)e^{-2\psi} - \frac{R}{2}e^{-\psi}, \tag{5.2}$$

with R being the inverse function of

$$\psi = \ln\left[2\mathcal{L}'(R)\right]. \tag{5.3}$$

The metric in the transformed Lagrangian (5.1) is

$$\tilde{g}_{ab} = e^{\psi} g_{ab}. \tag{5.4}$$

For our particular Lagrangian (2.9) we have from (5.3):

$$\psi = -\frac{3}{2}\ln\left(1+l^4R^2\right). \tag{5.5}$$

Now we restrict our attention to the range $R > 0$ where $\psi < 0$; the other sign can be treated analogously. Then (5.5) is inverted to

$$R = \frac{1}{l^2}\sqrt{e^{-2\psi/3} - 1}, \tag{5.6}$$

such that (5.2) becomes

$$V(\psi) = \frac{1}{2l^2}(e^{-5\psi/3} - e^{-\psi})\sqrt{e^{-2\psi/3} - 1}. \tag{5.7}$$

In the range under consideration, this is a positive and monotonously increasing function of $-\psi$ (see Fig. 5), with the large-ϕ behavior

$$V = \frac{1}{2l^2}e^{-2\psi}. \tag{5.8}$$

This is the typical exponential potential for power-law inflation. As mentioned at the end of Section 2, no exact de Sitter inflation exists. For $\psi \to 0$, also $V(\psi) \to 0$ like $4\sqrt{2/3}\psi^{3/2}$.

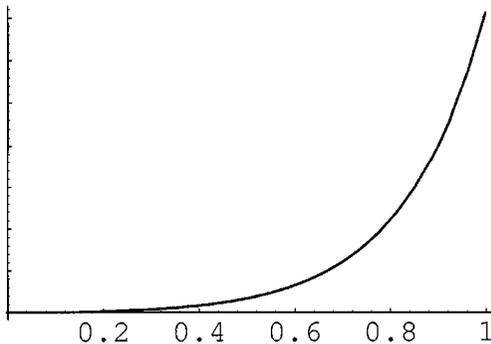**Figure 5.** Potential $V(\psi)$ associated with curvature-saturated action via Bicknell's theorem.

If $V(\psi)$ has a quadratic minimum at some ψ_0 with positive value $V_0 = V(\psi_0)$, then there exists a stable de Sitter inflationary phase. As a pleasant feature, the potential $V(\psi)$ has no maximum which have given rise to tachyons.

From Eq. (5.5) one can see that for weak fields, $\psi \sim R^2$, whereas a $R + R^2$-theory has $\psi \sim R$. In other words: In our model it is a better approximation to assume the conformal factor e^ψ to be approximately constant for weak fields than in $R + R^2$-theories, since at the level keeping only terms linear in R the two metrics g_{ab} and $\tilde{g}_{ab}$ in (5.4) coincide.

6. FRIEDMANN MODELS IN CURVATURE TIME

The expanding spatially flat Friedmann model may be parametrized with the help of *curvature time $a > 0$* as follows:

$$ds^2 = a^2 \left[\frac{da^2}{B(a)} - dx^2 - dy^2 - dz^2 \right], \tag{6.1}$$

where $B(a)$ is an arbitrary positive function determining R as

$$R = -\frac{3}{a^3} \frac{dB}{da}, \tag{6.2}$$

depending only on the first derivative of $B(a)$. This is a special feature of (6.1) since, in general, the curvature scalar depends on the second derivative of the metric components. Note also the *linear* dependence of R on $B' \equiv dB/da$, in contrast to the usual nonlinear dependence of the curvature scalar on the first derivative of the metric coefficients.

Let us recall some facts on Friedmann models in curvature time and exhibit the corresponding transformation to synchronized time.

6.1. From Curvature Time to Synchronized Time

The spatially flat Friedmann model in synchronized time has the metric

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \tag{6.3}$$

Metric (6.1) goes over to metric (6.3) via

$$dt = \frac{a\,da}{\sqrt{B(a)}}, \tag{6.4}$$

such that

$$t = t(a) = \int \frac{a\,da}{\sqrt{B(a)}}. \tag{6.5}$$

The inverse function $a(t)$ provides us with the desired transformation.

6.2. From Synchronized Time to Curvature Time

Consider $a(t)$ in an expanding model with

$$\dot{a} \equiv \frac{da}{dt} > 0. \tag{6.6}$$

Then we can invert $a(t)$ to $t(a)$, and have

$$B(a) = a^2 \left[\dot{a}(t(a))\right]^2 . \tag{6.7}$$

From this relation we understand why R depends on the first derivative of B only: B itself contains a derivative of a, and R is known to contain up to second order derivatives of $a(t)$.

6.3. Examples

Let $a(t) = t^n$, i.e., $t = a^{1/n}$, $\dot{a}(t) = nt^{n-1}$, $\dot{a}(t(a)) = na^{1-1/n}$. Then Eq. (6.7) yields

$$B(a) = n^2 a^{4-2/n}. \tag{6.8}$$

Let further $a(t) = e^{Ht}$, $H = \text{const.} > 0$, $\dot{a} = Ha$. Then

$$B(a) = H^2 a^4. \tag{6.9}$$

Obviously, Eq. (6.9) is a limiting form of Eq. (6.8) for $n \to \infty$. Equation (6.1) with $B(a)$ from (6.9) represents a vacuum solution of Einstein's theory with Λ-term where $\Lambda = 3H^2$, namely the de Sitter spacetime.

Let us also give some examples for the direct use the curvature time:

1. From Eq. (6.2) we see that $R = 0$ implies $B \equiv \text{const.}$, corresponding to $n = \frac{1}{2}$ in Eq. (6.8), i.e., $a = t^{1/2}$ in synchronized time. This is the usual Friedmann radiation model.
2. Also from Eq. (6.2), a constant $R \neq 0$ implies $B = C_1 + C_2 a^4$ with constants C_1 and C_2, $C_2 \neq 0$. For $C_1 = 0$, $C_2 = H^2$, this represents the de Sitter spacetime Eq. (6.9).
3. The dust-model in synchronized coordinates is given by $a = t^{2/3}$, i.e., with Eq. (6.8) we get

$$B(a) = \frac{4}{9}a, \tag{6.10}$$

such that $B' = \text{const.}$ Together with Eq. (6.2), this leads to

$$R\,a^3 = \text{const,} \tag{6.11}$$

ensuring mass conservation, because R is proportional to the mass density, and the pressure is negligible for dust.

6.4. The Variational Derivative

For the metric (6.1) we have

$$\sqrt{-g} \equiv \sqrt{-\det g_{ij}} = \frac{a^4}{\sqrt{B}}. \tag{6.12}$$

The Lagrangian for Einstein's theory with Λ-term reads

$$\mathcal{L} = (R + 2\Lambda)\sqrt{-g}. \tag{6.13}$$

With (6.2) and (6.7) we get from (6.13)

$$\mathcal{L} = \left(2\Lambda - \frac{3B'}{a^3}\right) a^4 B^{-1/2}. \tag{6.14}$$

The vanishing of the variational derivative

$$\frac{\delta \mathcal{L}}{\delta B} \equiv \frac{\partial \mathcal{L}}{\partial B} - \left(\frac{\partial \mathcal{L}}{\partial B'}\right)' = 0 \tag{6.15}$$

gives $B = H^2 a^4$ with $\Lambda = 3H^2$, i.e., the usual de Sitter spacetime. No integration is necessary, since the derivative of B cancels. Intermediate expressions are

$$\frac{\partial \mathcal{L}}{\partial B} = \left(2\Lambda - \frac{3B'}{a^3}\right) a^4 \left(-\frac{1}{2}\right) B^{-3/2}, \tag{6.16}$$

$$\frac{\partial \mathcal{L}}{\partial B'} = -3a B^{-1/2}, \quad \left(\frac{\partial \mathcal{L}}{\partial B'}\right)' = -3B^{-1/2} + \frac{3}{2} a B' B^{-3/2}. \tag{6.17}$$

6.5. Remaining Coordinate-Freedom

Translations in t do not change the form of the metric (6.3). This freedom is related to the fact that the integration constant in the integral (6.5) remains undetermined; this coordinate freedom has no analog in the metric in curvature time Eq. (6.1).

The metric (6.1) has the following property: It remains unchanged under multiplication of a^4 and B by the same positive constant. Such a constant factor appears if we multiply the spatial coordinates by a constant factor. In synchronized coordinates this property means that not a itself, but only the Hubble parameter

$$H(t) := \frac{\dot{a}}{a} \tag{6.18}$$

has an invariant meaning. By the same token, not $B(a)$ itself, but only $B(a)/a^4$ has an invariant meaning. In fact, from Eq. (6.7) we see that

$$\frac{B}{a^4} = H^2. \tag{6.19}$$

7. COSMOLOGICAL SOLUTIONS

Here we recall some formulas of Ref. [5], and present some new results for the curvature-saturated Lagrangian.

7.1. Solutions for Lagrangian R^m

For the Lagrangian $\mathcal{L} = R^m$, we obtain the following exact solutions for a closed Friedmann universe:

$$ds^2 = dt^2 - \frac{t^2}{2m^2 - 2m - 1} d\sigma_{(+)}^2, \tag{7.1}$$

where $d\sigma_{(+)}^2$ is the metric of the unit 3-sphere.

Analogously, for the open model

$$ds^2 = dt^2 - \frac{t^2}{2m - 2m^2 + 1} d\sigma_{(+)}^2. \tag{7.2}$$

Of course, both expressions are valid for positive denominators only.

For the spatially flat Friedmann model, it proves useful to employ the cosmic scale factor a itself as a time-like coordinate.

$$ds^2 = a^2[Q^2(a)da^2 - dx^2 - dy^2 - dz^2]. \tag{7.3}$$

This coordinate is meaningful as long as the Hubble parameter is different from zero, so that we cover only time intervals where the universe is either expanding or contracting. Possibly existing maxima or minima of the cosmic scale factor as seen in synchronized time can, however, been dealt by a suitable limiting process and patching. The curvature scalar reads now

$$R = \frac{6}{a^3 Q^3} \frac{dQ}{da}, \tag{7.4}$$

and to reduce the order of the field equation it proves useful to define

$$P(a) = \frac{d \ln Q}{da}. \tag{7.5}$$

Then the field equation is fulfilled if

$$0 = m(m - 1)\frac{dP}{da} + (m - 1)(1 - 2m)P^2 + m(4 - 3m)\frac{P}{a}. \tag{7.6}$$

Therefore, the spatially flat Friedmann models can be solved in closed form, but not always in synchronized coordinates.

7.2. Solutions for Lagrangian $\mathcal{L}_{cs}$

In the context of our curvature-saturated model, we shall restrict ourselves to the expanding spatially flat Friedmann model. The field equation written in synchronized or conformal time—the two most often used time coordinates used for this purpose—have the disadvantage that the number of terms is quite large, and that even in the simplest case $\mathcal{L} = \frac{1}{2}R^2$ we cannot give closed-form solutions, apart from the trivial solutions $R \equiv 0$ having the same geometry as the radiation universe ($a = \sqrt{t}$ in synchronized time t) and the de Sitter universe ($a = e^t$ in synchronized time t). So, we prefer to work in the less popular coordinates (7.3). In principle, the field equation should be of fourth order, but we shall reduce it to second order.

To find the field equation for a spatially flat Friedmann model with our Lagrangian, it is useful to consider first a general nonlinear Lagrangian and specialize to $\mathcal{L}_{cs}$ afterwards. To simplify (7.4), we define instead of $Q(a)$ the function $B(a) = Q(a)^{-2} > 0$ as a new dependent function. Then (7.3) reads

$$ds^2 = a^2 \left[\frac{da^2}{B(a)} - dx^2 - dy^2 - dz^2 \right] \tag{7.7}$$

and (7.4) goes over to

$$R = -\frac{3}{a^3} \frac{dB}{da}. \tag{7.8}$$

Thus, B itself does not appear explicitly, and only first, and not second derivatives are present. The geometric origin of this property is the same as in Schwarzschild coordinates—one integration constant is lost in the definition of the coordinates, and this makes curvature depend only on the first derivative of the metric.

From the 10 vacuum field equations (2.10) only the 00-component is essential; it is the constraint equation, therefore it has one order less than the full field equation, but if the constraint is fulfilled always, then all other components are fulfilled, too.[3] Together with Eq. (7.8) we should now expect that the fourth order field equation (2.10) can be reduced to one single second order equation for $B(a)$, where hopefully, B itself no more appears.

The equation $\Theta_{00} = 0$ is via (2.4) and (2.8) equivalent to

$$0 = 3\mathcal{L}' \left(2B - a\frac{dB}{da} \right) - a^4 \mathcal{L} - 18aB\,\mathcal{L}'' \frac{d}{da} \left(\frac{1}{a^3} \frac{dB}{da} \right), \tag{7.9}$$

[3] This behavior is known already from the Friedmann equation in General Relativity: Energy density is proportional to the square of the Hubble parameter which contains only a first derivative.

which is much simpler than the analogous equation in synchronous time, as observed here for the first time.

Before we insert our Lagrangian $\mathcal{L}_{cs}$ into (7.9), let us cross check its validity by solving known problems: If $\mathcal{L}''$ vanishes identically, then $\mathcal{L}'$ is a constant, and we return to Einstein's theory. The case $B \equiv$ const. gives the radiation universe, while $B = a^4$ is the exact de Sitter solution. For the Lagrangian $\mathcal{L} = \frac{1}{2}R^2$ with $\mathcal{L}' = R$ and $\mathcal{L}'' = 1$, and Eq. (7.9) reduces to

$$0 = a\dot{B}^2 - 4aB\ddot{B} + 8B\dot{B}, \tag{7.10}$$

where a dot denotes differentiation with respect to a. Again, $B = a^4$ is the exact de Sitter solution. Defining $\beta = \ln B$ and $z = a\dot{\beta}$, Eq. (7.10) goes over in

$$4a\dot{z} = 3z(4 - z). \tag{7.11}$$

With $\alpha = \ln a$ we arrive at

$$4\frac{dz}{d\alpha} = 3z(4 - z), \tag{7.12}$$

which can be solved in closed form. Qualitatively it is clear that $z = 4$, i.e., the de Sitter solution, represents an attractor. Solving Eq. (7.12) we obtain in the region $0 < z < 4$:

$$z = 2 + 2\tanh\left(\frac{3}{2}\alpha\right), \tag{7.13}$$

showing explicitly that $z \to 4$ for $\alpha \to \infty$. The metric can be calculated from

$$\dot{\beta} = \frac{2}{a}\left(1 + \frac{a^3 - 1}{a^3 + 1}\right), \tag{7.14}$$

using the identity

$$\tanh \ln x = \frac{x^2 - 1}{x^2 + 1}. \tag{7.15}$$

After these preparations we are ready to deal with our Lagrangian $\mathcal{L}_{cs}$. We insert $\mathcal{L}$ and $\mathcal{L}'$ from Eq. (4.4), and

$$\mathcal{L}'' = -3l^4R(1 + l^4R^2)^{-5/2} \tag{7.16}$$

into Eq. (7.9) and obtain, after setting $l = 1$, the simple expression

$$54a^9B\dot{B}\frac{d}{da}(a^{-3}\dot{B}) = a^5(a^6 + 9\dot{B}^2)(2B - a\dot{B}) + \dot{B}(a^6 + 9\dot{B}^2)^2. \tag{7.17}$$

In these coordinates, the flat Minkowski spacetime does not exist, and the radiation universe $R = 0$ is not a solution. This is why $B =$ const. yields no solution to Eq. (7.17). Also, as was known from the beginning: the de Sitter spacetime $B = a^4$ is not an exact solution here. However, in the nearby-region

where the Lagrangian is well approximated by a quadratic function in R with a nonvanishing linear term, the behavior of the solutions is quite similar to that of $R + R^2$-models, where no exact de Sitter solution exists, but a quasi de Sitter solution represents a transient attractor with sufficient long duration to solve the known cosmological problems. These calculations have been presented at different places, most explicitly in Ref. [6]. After this phase, the universe goes to the weak-field behavior, where our model behaves as usual.

The main departure of our model from the usual one is in the region of large curvature scalar, where $|\dot{B}|$ is large compared to a^3. To find out the behavior of the solutions in this limit, we compare the leading terms in Eq. (7.7) and see that $\ddot{B}$ is proportional to $\dot{B}^4$, where the coefficient of proportionality is positive and slowly varying. Thus, we find approximately $B(a) \approx a^{2/3}$ for small a. This implies the existence of a big-bang singularity, but with a different behavior: From Eq. (7.7) we obtain

$$ds^2 = a^2 \left[\frac{da^2}{a^{2/3}} - dx^2 - dy^2 - dz^2 \right], \tag{7.18}$$

which corresponds in synchronized time to the behavior

$$ds^2 = dt^2 - t^{6/5}(dx^2 + dy^2 + dz^2), \tag{7.19}$$

this being a good approximation to the exact metric for small t, differing from the usual big-bang behavior in almost all other models. Further details of our model will be presented elsewhere.

7.3. The Cosmological Singularity

Here we present the argument with the singularity behaviour mentioned at the end of section 2: In our model, differently from Einstein's theory, the divergence of the curvature does not necessarily imply the divergence of any part of the energy-momentum-tensor. Let us concentrate on the trace. The r.h.s. of Eq. (2.11) reads

$$\frac{1}{8\pi G} \left\{ \frac{R + 2l^4 R^3}{(1 + l^4 R^2)^{3/2}} - 3\Box \left[\frac{1}{(1 + l^4 R^2)^{3/2}} \right] \right\}$$

and this expression must be equal to the trace T of the energy-momentum tensor. In Einstein's theory, $R \to \pm\infty$ necessarily implies $T \to \pm\infty$, whereas here, T may remain finite even if $R \to \infty$.

Detailed numerical calculations would support this qualitative picture, however, we postpone such calculations until we have a more strictly physically motivated form of the Lagrangian.

8. DISCUSSION

We have argued that the gravitational action $\mathcal{A}$ has a decreasing dependence on R for increasing $|R|$. Such a behavior is expected from the spacetime stiffness caused by the vacuum fluctuations of all quantum fields in the universe.

Our model does not have the tachyonic disease of $R + R^2$ models studies by Stelle [17] and others [18].

Since our model has an action which interpolates between Einstein's action and a pure cosmological term, it promises to have interesting observable consequences which may explain some of the experimental cosmological data.

The heat-kernel expansion of the effective action in a curved background is closely related to the Seeley-Gilkey coefficients [19], and for higher loop expansion also higher powers of curvature appear: To get the n-loop approximation one has to add terms until $\sim R^{n+1}$, a behavior which also happens in the string effective action [20]. So, if one cuts this procedure at a certain value of n, one gets always as leading term for high curvature values a term like $\sim R^{n+1}$. However, the n-loop approximations need not converge to the correct result if one simply takes $n \to \infty$ in the n-loop-result. In fact, what we have used in the present paper is such an example:

$$\mathcal{L}_{\text{cs}} = \frac{R}{16\pi G \sqrt{1 + l^4 R^2}} = \frac{R}{16\pi G} + \sum_{k=1}^{\infty} b_k R^{2k+1} \tag{8.1}$$

with some real constants b_k, where

$$b_1 = -\frac{l^4}{32\pi G} \tag{8.2}$$

but the Taylor expansion on the right hand side diverges for $R > l^{-2}$. So, the Taylor expansion is useful for small R-values only, and for large values R we need a correct analytical continuation.

Prigogine et al. have proposed in Eq. (18) of Ref. [21] a model where the effective gravitational constant depends on the Hubble parameter of a Friedmann model. Though this ansatz depends on the special $3 + 1$-decomposition of spacetime, it shares some similarities with the model discussed here. More recent developments how to find a well-founded gravitational action from considering quantum effects can been found in [22] and [23].

Quite recently, see for instance [26], accelerated expansion models of the universe have been discussed and compared with new observations. We postpone the comparison of our model with these observations to later work. A continuation of the present paper is [30], where the inclusion of matter is explicitly done, and thus the present qualitative results are substantiated.

APPENDIX A: NEWTONIAN LIMIT IN A NONFLAT BACKGROUND

The Newtonian limit of a theory of gravity is defined as follows: It is the weak-field slow-motion limit for fields whose energy-momentum tensor is dominated by its zero-zero component in comoving time. Usually, the limit is formed in a flat background, and sometimes, this is assumed to be a necessary assumption. This is, however, not true, and we show here briefly how to calculate the Newtonian limit in a nonflat background, Moreover, our approach is different from what is usually called Newtonian cosmology. To have a concrete example, we take the background as a de Sitter spacetime. Recent progress for calculating the Newtonian limit in the presence of an effective cosmological constant can be seen in [31].

The slow-motion assumption allows us to work with static spacetime and the matter, assuming the energy-momentum tensor to be

$$T_{ij} = \rho \, \delta_i^0 \delta_j^0, \tag{A1}$$

where ρ is the energy density, and time is assumed to be synchronized. The de Sitter spacetime in its static form can be given as

$$ds^2 = -(1 - kr^2)dt^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2, \tag{A2}$$

where $x^0 = t$, $x^1 = r$, $x^2 = \chi$, $x^3 = \theta$ and $d\Omega^2 = d\chi^2 + \sin^2 \chi \, d\theta^2$ is the metric of the 2-sphere. In this Appendix, we have changed the signature of the metric from $(+ - -)$, which is usual in cosmology, to $(- + +)$, which leads to the standard definition of the Laplacian.

The parameter k characterizes the following physical situations: For $k = 0$, we have the usual flat background. By setting $k = 0$ we can therefore compare the results with the well-known ones. The case $k > 0$ corresponds to a positive cosmological constant Λ. In the calculations, we must observe that the time coordinate t fails to be a synchronized for $k \neq 0$, but it is obvious from the context how to obtain the synchronized time from it.

In the coordinates (A2), there is a horizon at $r = r_0 \equiv \frac{1}{\sqrt{k}}$. So, our approach makes sense in the interval $0 < r < r_0$. However, r_0 shall be quite large in comparison with the system under consideration, so that we do not meet a problem here.

Now, the following ansatz seems appropriate:

$$ds^2 = -(1 - kr^2)(1 - 2\varphi)dt^2 + \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)(1 + 2\psi), \tag{A3}$$

where φ and ψ depend on the spatial coordinates only. The weak-field assumption allows us to make linearization with respect to φ and ψ. An extended matter configuration can be obtained by superposition of point particles, so we only need to solve the problem for a δ-source at $r = 0$. This one is spherically symmetric, so

we may assume $\varphi = \varphi(r)$ and $\psi = \psi(r)$ in Eq. (A3). For the metric components we get:

$$g_{00} = -(1 - kr^2)(1 - 2\varphi), \quad g_{11} = \frac{1 + 2\psi}{1 - kr^2}, \quad g_{22} = r^2(1 + 2\psi),$$

$$g_{33} = g_{22} \cdot \sin^2 \chi. \tag{A4}$$

The inverted components are up to linear order in φ and ψ:

$$g^{00} = -\frac{1 + 2\varphi}{1 - kr^2}, \quad g^{11} = (1 - kr^2)(1 - 2\psi), \quad g^{22} = \frac{1 - 2\psi}{r^2},$$

$$g^{33} = g^{22} \sin^{-2} \chi, \tag{A5}$$

which gives the Christoffel symbols

$$\Gamma^0_{01} = -\varphi' - \frac{kr}{1 - kr^2}, \tag{A6}$$

$$\Gamma^1_{00} = (1 - kr^2)[-kr + 2kr(\varphi + \psi) - \varphi'(1 - kr^2)], \tag{A7}$$

$$\Gamma^1_{11} = \psi' + \frac{kr}{1 - kr^2}, \tag{A8}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \psi' + \frac{1}{r}, \tag{A9}$$

$$\Gamma^1_{22} = -r(1 - kr^2) - \psi' r^2(1 - kr^2), \tag{A10}$$

$$\Gamma^1_{33} = \sin^2 \chi \, \Gamma^1_{22}, \tag{A11}$$

$$\Gamma^3_{32} = \cot \chi, \tag{A12}$$

$$\Gamma^2_{33} = -\sin \chi \cos \chi, \tag{A13}$$

and the Ricci tensor reads

$$R_{00} = -3k(1 - kr^2) - \varphi''(1 - kr^2)^2 - \frac{2\varphi'}{r}(1 - kr^2) + 6k(\varphi + \psi)(1 - kr^2)$$

$$+ kr(1 - kr^2)(5\varphi' - \psi'), \tag{A14}$$

$$R_{11} = -2\psi'' + \varphi'' - \frac{2}{r}\psi' + \frac{3k}{1 - kr^2} + \frac{kr}{1 - kr^2}(\psi' - 3\varphi'), \tag{A15}$$

$$R_{22} = 3kr^2 - \psi'' r^2(1 - kr^2) - \psi'(2r - 4kr^3) + (\varphi' - \psi')(r - kr^3), \tag{A16}$$

$$R_{33} = R_{22} \cdot \sin^2 \chi. \tag{A17}$$

Before we discuss these equations, we consider two obvious limits:

For $k = 0$, we see that $R_{00} = -\varphi'' - 2\varphi'/r = -\Delta\varphi$, leading to the usual Newtonian limit $\Delta\varphi = -4\pi G\rho$.

For $\varphi = \psi = 0$ we get for the Ricci tensor:

$$R_0^0 = R_1^1 = R_2^2 = R_3^3 = 3k, \tag{A18}$$

and thus the de Sitter spacetime with $R = 12k$ for $k > 0$.

Returning to the general case we have

$$\frac{R}{2} = 6k - 12k\psi + (\varphi'' - 2\psi'')(1 - kr^2) + \frac{2}{r}\varphi' - 5kr\varphi' - \frac{4}{r}\psi' + 7kr\psi' \tag{A19}$$

and then

$$R_0^0 - \frac{R}{2} = -3k + 6k\psi + 2\psi''(1 - kr^2) - 6kr\psi' + \frac{4}{r}\psi'. \tag{A20}$$

The other components have a similar structure and can be calculated easily from the above equations. The first term of the r.h.s., $-3k$, will be compensated by the Λ-term. The usual gauging to $\psi \to 0$ and $\varphi \to 0$ as $r \to \infty$ is no more possible because for $r > r_0$ our approximation is no more valid. As an alternative gauge we add such constant values to ψ and φ that they are approximately zero in the region under consideration. So we may disregard the term $6k\psi$. All remaining terms with k can be obtained from those without k by multiplying with factors of the type $1 + \epsilon$ where $\epsilon \approx kr^2$, $k = 1/r_0^2$, with r_0 being of the order of magnitude of the world radius. In a first approximation, this gives only a small correction to the gravitational constant. In a second approximation, there are deviations from the $1/r$-behavior.

An analogous discussion for the Lagrangian $R + l^2 R^2$ tells us that in a range where $l \ll r \ll r_0$, the potential behaves like $(1 - c_1 e^{-r/l})/r$, as in flat space.

APPENDIX B: THE ABSENCE OF GHOSTS AND TACHYONS

Here we show in more details what has been stated after Eq. (3.3). In the conformally transformed picture with a scalar field, the absence of tachyons (i.e., particles with wrong sign in front of the potential term) becomes clear from the form of the potential. For checking ghosts (i.e., particles with wrong sign in front of the kinetic term) we have to go a little more into the details: In Stelle [27] the particle content of fourth order gravity with terms up to quadratic order has been determined, and the existence/absence of ghosts and tachyons has been given in dependence on the free constants of the theory. In the first of Refs. [4], the analogous calculation as in [27] has been done for a term R^3 added to the Einstein–Hilbert-Lagragian. Let us give here the argument for general $n \geq 3$: If R^n is in $\mathcal{L}$, then the term R^{n-1} and its derivatives are in the corresponding expression after variational derivative with respect to the metric. In the result, all terms represent products of at least $n - 1$ small quantities; because of $n \geq 3$ these are always at

least two factors; thus, they all vanish in the linearization about the Minkowski space–time.

Now, one might be tempted to require the analogous linearization properties for a Friedmann–Robertson–Walker background. However, linearization around other than flat space–times is not at all a trivial task, see [28], even for Einstein's theory: For the closed Friedmann model, Einstein's theory is linearization unstable, for spatially flat models it is stable, and for the open Friedmann model the result is – contrary to other claims in the older literature – not yet known. We face the further problem that linearization around the de Sitter space-time is complicated to determine, because the same geometry can be locally represented as a spatially flat as well as a closed Friedmann model. So, we leave the question of linearization stability with non-flat background of our model unanswered.

Another type of reasoning was given quite recently: In [29] the possibility has been discussed that the contributions to the Lagrangian coming of gravitons on the one hand and of gravitinos on the other may cancel each other to avoid the ghost problem.

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REFERENCES

[1] A. Sakharov, (1967). *Dokl. Akad. Nauk SSSR* **177**, 70; *Vacuum quantum fluctuations in curved space and the theory of gravitation*, reprinted in (2000). *Gen. Rel. Grav.* **32**, 365. See also the editor's note on that paper in H.-J. Schmidt, (2000). *Gen. Rel. Grav.* **32**, 361.

[2] S. Adler, (1982). *Rev. Mod. Phys.* **54**, 729.

[3] See, for example J. Hwang, H. Noh, (2000). *Phys. Rev. D* **61**, 043511, *Conserved cosmological structures in the one-loop superstring effective action* (astro-ph/9909480); G. Ellis, D. Roberts, D. Solomons, and P. Dunsby, gr-qc/9912005, to appear in *Phys. Rev. D*.

[4] H.-J. Schmidt, *Comparing selfinteracting scalar fields and $R + R^3$ cosmological models*, gr-qc/0106035, (1987). *Astron. Nachr.* **308**, 183; H.-J. Schmidt, *On the critical value of the curvature scalar*, (1986). *Proc. Conf. General Relativity GR* **11**, Stockholm, p. 117.

[5] H.-J. Schmidt, *New exact solutions for power-law inflation Friedmann models*, gr-qc/0109004, (1990). *Astron. Nachr.* **311**, 165; (1993). *Gen. Rel. Grav.* **25**, 87, 863.

[6] V. Müller and H.-J. Schmidt, (1985). *Gen. Rel. Grav.* **17**, 769 and 971.

[7] H.-J. Schmidt, *Stability and Hamiltonian formulation of higher derivative theories*, (1994). *Phys. Rev. D* **49**, 6354; (1996). *Phys. Rev. D* **54**, 7906; gr-qc/9404038. See also H.-J. Schmidt, (1994). *Phys. Rev. D* **50**, 5452; (1995). *Phys. Rev. D* **52**, 6198.

[8] H.-J. Schmidt, *A new duality transformation for fourth-order gravity*, (1997). *Gen. Rel. Grav.* **29**, 859.

[9] S. Hawking, T. Hertog, and H. Reall, (2000). *Phys. Rev. D* **62**, 043501.

[10] A. Starobinsky, (1980). *Phys. Lett. B* **91**, 99; A. Starobinsky and H.-J. Schmidt, (1987). *Class. Quant. Grav.* **4**, 695.

[11] V. Sahni and A. Starobinsky, (2000). *Int. J. Mod. Phys. D* **9**, 373–443.

[12] H. Kleinert and A. Zhuk; unpublished Preprint 1993; *Finite-Size and Temperature properties of Matter and Radiation fluctuations in Closed Friedmann Universe*; H. Kleinert and A. Zhuk, (1996). *Theor. Math. Phys.* **109**, 307; *Casimir effect at nonzero temperature in closed Friedmann universe*; the files can be downloaded from http://www.physik.fu-berlin.de/~kleinert/as publications Nr. 218 and 251 resp., See also the summary given at: (2000). *Zentralblatt MATH* **933**, 674.

[13] V. Gurovich, (1971). *Sov. Phys. Dokl.* **15**, 1105.

[14] V. Gurovich, I. Tokareva, (1999). *Gen. Rel. Grav.* **31**, 21.

[15] J. Fabris, S. Reuter, *Quantum cosmology in higher derivative and scalar-tensor gravity*, (2000). *Gen. Rel. Grav.* **32**, 1345.

[16] Y. Anini, p. 183 in: *Current topics in mathematical cosmology*, Eds.: M. Rainer, H.-J. Schmidt (WSPC Singapore 1998); R. Brandenberger, V. Mukhanov, A. Sornborger, (1993). *Phys. Rev. D* **48**, 1629.

[17] K. Stelle, (1977). *Phys. Rev. D* **16**, 953; S. Weinberg, in: *General Relativity*, Eds. S. Hawking, W. Israel, Cambridge University Press 1979. The latter paper claims that the results of Stelle were already presented at conferences by S. Weinberg in 1974 and by S. Deser in 1975.

[18] L. Buchbinder, S. Odintsov, I. Shapiro, (1992). *Effective action in quantum gravity*, IOP Bristol.

[19] P. Gilkey, (1975). *J. Diff. Geom.* **10**, 601; I. Avramidi, (2000). *Heat Kernel and Quantum Gravity*, Springer Berlin.

[20] I. Jack, D. Jones, N. Mohammedi, (1989). *Nucl. Phys. B* **322**, 431.

[21] I. Prigogine, J. Geheniau, E. Gunzig, P. Nardone, (1989). *Gen. Rel. Grav.* **21**, 767.

[22] G. Siemieniec-Ozieblo, A. Woszczyna, *Order-reduction in semiclassical cosmology*, (1999). *Phys. Rev.* **D 59**, 083504.

[23] L. Parker, A. Raval, *Nonperturbative effects of vacuum energy on the recent expansion of the universe*, (1999). *Phys. Rev. D* **60**, 063512.

[24] H. Kleinert, (2000). *Gen. Rel. Grav.* **32**, 769.

[25] G. Bicknell, (1974). *J. Phys. A* **7**, 341, 1061. The original papers did not attract much attention when they were published. A decade later the results were rediscovered simultaneously by several researchers and have since enjoyed many applications, for instance in Ref. [15]; in Chapter 6 of H. v. Elst, Thesis 1996, Queen Mary and Westfield College London; S. Capozziello, G. Lambiase, *Higher-order corrections to the effective gravitational action from Noether symmetry approach*, (2000). *Gen. Rel. Grav.* **32**, 295; L. Amendola, *Scaling solutions in general nonminimal coupling theories*, (1999). *Phys. Rev. D* **60**, 043501, and D. Barraco, E. Dominguez, R. Guibert, *Conservation laws, symmetry properties, and the equivalence principle in a class of alternative theories of gravity*, (1999). *Phys. Rev. D* **60**, 044012. The authors of these papers attribute this theorem to several different persons.

[26] Yi-Ping Qin, Preprint gr-qc/0005054.

[27] K. Stelle, (1978). *Gen. Rel. Grav.* **9**, 353.

[28] L. Bruna and J. Girbau, *Linearization stability of the Einstein equation for Robertson-Walker models.* (1999). *J. Math. Phys.* **40**, 5117.

[29] D. Anselmi, (2000). *Class. Quant. Grav.* **17**, 2847..

[30] S. Capozziello, G. Lambiase, H.-J. Schmidt, gr-qc/0101090, p. 697 in "Fluctuating Paths and Fields," WSPC Singapore 2001, Eds: W. Janke, A. Pelster, H.-J. Schmidt, M. Bachmann.

[31] M. Nowakowski, (2001). *Int. J. Mod. Phys. D* **10**, 649.