Three-Loop Ground-State Energy of O(N)-Symmetric Ginzburg-Landau Theory above T_c in $4-\varepsilon$ dimensions with minimal subtraction

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As a step towards deriving universal amplitude ratios for the superconductive phase transition we calculate the vacuum energy density in the symmetric phase of O(N)-symmetric scalar QED in $D = 4-\varepsilon$ dimensions in an ε-expansion using the minimal subtraction scheme commonly denoted by $\overline{\text{MS}}$. From the diverging parts of the diagrams, we obtain the renormalization constant of the vacuum Z_v which also contains information on the critical exponent α of the specific heat. As a side result, we use an earlier two-loop calculation of the effective potential [15] to determine renormalization constant of the scalar field Z_ϕ up to two loops.

I. INTRODUCTION

One of the most intriguing problems in the physics of critical phenomena is a theoretical understanding of the superconductive phase transition within the renormalization group approach. A first discussion was given in 1974 by Halperin, Lubensky, and Ma [1] on the basis of the Ginzburg-Landau, or $U(1)$ Abelian-Higgs model, in $4-\epsilon$ dimensions, generalizing a similar four-dimensional analysis of Coleman and Weinberg [2]. In a one-loop approximation, they did not find an infrared-stable fixed point, and in spite of much effort it is still unclear whether a higher-loop renormalization group analysis would be capable of explaining the existence of a critical point in three dimensions. Experimentally, this existence has been confirmed only recently with the advent of high-T_c superconductors. In conventional superconductors, the Ginzburg criterion [3], or the more relevant criterion for the size of phase fluctuations [4], predicted a too small temperature interval for the critical regime to see anything but mean-field behavior. Evidence had so far come only from Monte-Carlo simulations [5] and an analogy with smectic-nematic transitions in liquid crystal

[6]. Only by artificially allowing for an unphysically large number of replica n of the complex field ϕ, larger than 365, has it been possible to stabilize the renormalization flow. Historically, only a dual disorder formulation has brought some insight revealing the existence of a tricritical point at a Ginzburg parameter $\sqrt{2}\kappa \approx 0.77$, a material parameter characterizing the ratio between magnetic and coherence length scales [7,8].

The confusing situation in the Ginzburg-Landau model certainly requires further investigation in higher loop approximations. So far, two loop renormalization group calculations in $4-\varepsilon$ dimensions have not yet produced satisfactory results [9,10]. Analyses in $d = 3$ dimensions a la Parisi have also left many open questions [11]. An interesting observation was made by Nogueira [12], that an anomalous momentum instability below T_c may be responsible for the unusual resistance of the superconductive transition to theory. Hope for a better understanding has also been raised by a recent renormalization group study in $d = 3$ dimensions performed for the first time *below* T_c where a fixed point has been found at the one-loop level [13].

Once an infrared-stable fixed point is located, it will determine critical exponents and amplitude ratios. While the former can be extracted from the perturbation expansions of the renormalization constants, the latter requires knowledge of the singularities in powers of $1/\varepsilon$ of the vacuum energy, which is written for dimensional reasons as $m^4 Z_v/g\mu^\varepsilon$. As usual, μ is the mass scale of dimensional regularization, and m, g are renormalized mass and coupling constant of the complex order field ϕ. The singularities are collected in the dimensionless renormalization constant Z_v. This will be done with the so-called modified minimal subtraction scheme, denoted by $\overline{\rm MS}$ [14].

The calculation of this constant up to three loops will be the central result of this paper. During the calculations, we also recover information about the other renormalization constants of the theory.

Since the theory has so far only a fixed point for large $n > 365$, we shall keep an arbitrary number of replica in the theory, the physical case being $n = 1$. The n complex fields are coupled minimally to an Abelian gauge field which describes magnetism. The $N = 2n$ real and imaginary parts of the fields are assumed to have an O(N)-symmetric quartic self-interaction.

II. MODEL

The Lagrangian density to be studied contains $n = N/2$ complex scalar fields ϕ_B coupled to the magnetic vector potential $A_{B\mu}$ and reads, with a covariant gauge fixing,

$$\mathcal{L} = |D_{B\mu}\phi_B|^2 + m_B^2\phi_B^2 + \frac{g_B}{4}(\phi_B^2)^2 + \frac{1}{4}F_{B\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_{B\mu})^2, \tag{1}$$

where $D_{B\mu} = \partial_\mu - ie_B A_{B\mu}$ denotes the covariant derivative, $F_{B\mu\nu} = \partial_\mu A_{B\nu} - \partial_\nu A_{B\mu}$ is the field strength, and α a gauge parameter. The bare character is indicated by the subscript 'B'. In principle, there are also ghost fields which, however, decouple in the symmetric phase. and remain massless. Working in dimensional regularization they do not contribute to the energy because of Veltman's rule

$$\int \frac{d^D p}{(2\pi)^D}\, p^\alpha = 0 \quad \text{for all} \quad \alpha, \tag{2}$$

The coefficient 1/4 in front of the coupling constant g_B is conventional. The Feynman diagrams associated with the vacuum energy of the Lagrangian (1) have been generated iteratively in Ref. [16]. At some places it will be useful to compare our results with those of an earlier work [15], we have derived the two-loop effective potential above and *below* T_c. For such comparisons, a replacement $g_B \to 2g_B/3$ is required.

A full extension of the work in [15] is a highly nontrivial since the effective potential requires the calculation of Feynman diagrams with three different masses. For this reason we shall restrict ourselves in this paper to the symmetric phase $T > T_c$, where the field expectations vanish and the system contains only two masses, which greatly simplifies the problem, in particular, since one of the masses, the photon mass, is zero. As a consequence, most diagrams can be reduced to scalar integrals which can be computed exactly. The only exception is the watermelon— or basketball—diagram whose ε-expansion is, however, known to sufficiently high order in ε [17].

As in [15], we shall use throughout Landau gauge, $\alpha \to 0$ which enforces a transverse photon field. This has the advantage of being infrared stable [18,19].

III. RENORMALIZATION

The renormalization constants of the model are defined by

$$\phi_B = \phi\sqrt{Z_\phi}, \quad A_{B\mu} = A_\mu\sqrt{Z_A}, \quad m_B^2 = m^2\frac{Z_{m^2}}{Z_\phi}, \quad g_B = g\mu^\varepsilon\frac{Z_g}{Z_\phi^2}, \quad e_B = e\mu^{\varepsilon/2}\frac{Z_e}{Z_\phi\sqrt{Z_A}} = \frac{e}{\sqrt{Z_A}}\mu^{\varepsilon/2}, \tag{3}$$

where, in the last equation, we have taken into account the relation $Z_e = Z_\phi$, which is a consequence of the Ward identity. Heuristically, this equality comes from the requirement that the covariant derivative $D_{B\mu}\phi_B$ should not only be invariant with respect to gauge transformations but also with respect to renormalization. Thus it must acquire the same normalization factor as the field itself, going over into $\sqrt{Z_\phi}D_\mu\phi$. An arbitrary mass scale μ in (3) serves to define dimensionless coupling constants g and e.

The above multiplicative renormalizations are not sufficient to extract all finite information from the theory. The vacuum energy requires a special treatment, as emphasized in previous work of one of the authors (BK) [20]. Dimensionality requires the effective potential to have mass dimension D. To have a finite vacuum energy we must add a to the Lagrangian a counterterm

$$E_v^c = \frac{m^4}{g\mu^\varepsilon} Z_v. \tag{4}$$

The different renormalization constants may be expanded in powers of the fluctuation size $\hbar$ as follows:

$$Z_j = 1 + \sum_{l=1}^{L} \left[\frac{\hbar}{(4\pi)^2}\right]^l Z_j^{(l)}, \qquad Z_v = \sum_{l=1}^{L} \left[\frac{\hbar}{(4\pi)^2}\right]^l Z_v^{(l)}, \tag{5}$$

where the subscript j stands for fields and coupling constants ϕ, A, m, g, e. In minimal subtraction, each expansion coefficient has simple power series

$$Z_j^{(l)} = \sum_{k=0}^{l} g^{l-k} e^{2l} \left(c_j^1 \varepsilon^{-l} + c_j^2 \varepsilon^{1-l} + \ldots + c_j^l \varepsilon^{-1}\right), \tag{6}$$

except for $Z_g^{(l)}$ where the systematics is

$$g Z_g^{(l)} = \sum_{k=0}^{l+1} g^{l+1-k} e^{2l} \left(c_j^1 \varepsilon^{-l} + c_j^2 \varepsilon^{1-l} + \ldots + c_j^l \varepsilon^{-1}\right), \tag{7}$$

Initially, one finds also pole terms of the form d $1/\varepsilon^2 \times \ln$, $1/\varepsilon \times \ln$, and $1/\varepsilon \times \ln^2$, where ln is short for $\ln(m^2/\bar{\mu}^2)$ with $\bar{\mu}$ being related to the mass scale μ via the Euler-Mascheroni constant γ_E as $\bar{\mu}^2 = 4\pi\mu^2 \exp(-\gamma_E)$. These, however, turn out to cancel each other, which provides us with a nice consistency check of the renormalization procedure [22].

IV. FEYNMAN RULES AND VACUUM DIAGRAMS

The elements of the Feynman diagrams associated with the Lagrangian (1) are

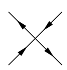
α β $$= \frac{\delta_{\alpha\beta}}{p^2 + m_B^2}, \tag{8}$$

(9)

μ ν $$= \frac{\delta_{\mu\nu} - p_\mu p_\nu/p^2}{p^2}, \tag{10}$$

(11)

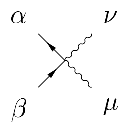
α δ



β γ

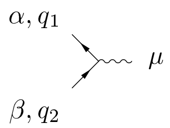
$$= -\frac{g_B}{2}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{12}$$

(13)



$$= -2e_B^2 \delta_{\alpha\beta}\delta_{\mu\nu}, \tag{14}$$

$$\tag{15}$$



$$= e_B \delta_{\alpha\beta}(q_1 + q_2)_\mu. \tag{16}$$

$$\tag{17}$$

In Ref. [16], the diagrams of the theory have been generated recursively up to four loops. Table 1 shows all diagrams needed for the three-loop vacuum energy above T_c. We have omitted those which contain massless separable loop integral which vanish by Veltman's rule (2).

V. $Z_v^{(3)}$ AND $Z_\phi^{(2)}$

The determination of the two-loop effective potential below T_c [15] allows to extract the following one-loop and two-loop contributions to the renormalization constants. Note that a factor $(4\pi)^{-2l}$ has been taken out in the definition (5).

$$Z_{m^2}^{(1)} = g\frac{(N+2)}{2\varepsilon}, \tag{18}$$

$$gZ_g^{(1)} = \frac{g^2(N+8)+48e^4}{2\varepsilon}, \tag{19}$$

$$Z_\phi^{(1)} = e^2\frac{6}{\varepsilon}, \tag{20}$$

$$Z_A^{(1)} = -e^2\frac{N}{3\varepsilon}, \tag{21}$$

$$Z_v^{(1)} = g\frac{N}{2\varepsilon}, \tag{22}$$

$$Z_{m^2}^{(2)} = \frac{(N+2)}{\varepsilon^2}\left[\frac{1}{4}g^2(N+5) - 3ge^2 + 6e^4\right] - \frac{1}{\varepsilon}\left[\frac{3}{8}g^2(N+2) - 2ge^2(N+2) - e^4(5N+1)\right], \tag{23}$$

$$\begin{aligned} gZ_g^{(2)} = &\frac{1}{\varepsilon^2}\left[\frac{1}{4}g^3(N+8)^2 - 3g^2e^2(N+8) + 12ge^4(N+8)^2 + 2e^6(N+18)\right] \\ &- \frac{1}{\varepsilon}\left[\frac{1}{4}g^3(5N+22) - 2g^2e^2(N+5) - 2ge^4(5N+13) + \frac{4}{3}e^6(7N+90)\right], \end{aligned} \tag{24}$$

$$Z_v^{(2)} = \frac{N}{\varepsilon^2}[-3ge^2 + \frac{1}{4}g^2(N+2)] + 2ge^2\frac{N}{\varepsilon}. \tag{25}$$

A. Results for the integrals

In this section, we give the value of the diagrams listed in Table 1. Although the exact value of part of the three-loop diagrams is known, for the sake of brevity we only give the ε expansion through order ε^0. The notation is as follows: I is the integral for the case $N=2$, omitting the weights of Table 1 and setting coupling constants and the scalar mass to unity. The first index is the loop order while the second one counts through the diagrams within each loop order in Table 1.

We have (with $D = 4 - \varepsilon$):

$$I_{1a} = -\frac{1}{(4\pi)^{D/2}}\Gamma(-D/2), \tag{26}$$

$$I_{2a} = \frac{1}{(4\pi)^D}\left[\frac{4\Gamma(2-D/2)}{3-D} - \Gamma(1-D/2)\right]\Gamma(1-D/2), \tag{27}$$

$$I_{2b} = \frac{1}{(4\pi)^D}\Gamma(1-D/2)^2, \tag{28}$$

$$\begin{aligned} I_{3a} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\Bigg[&\frac{4}{\varepsilon^2} + \frac{-\frac{29}{3} + 64\zeta(3)}{\varepsilon} - \frac{943}{12} + \frac{64\ln^4 2}{3} + 512\mathrm{Li}_4\left(\frac{1}{2}\right) \\ &+ \frac{3\zeta(2)}{2} - 128\zeta(2)\ln^2 2 + 288\zeta(3) - 352\zeta(4)\Bigg], \end{aligned} \tag{29}$$

$$I_{3b} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{96}{\varepsilon^3} + \frac{242}{\varepsilon^2} + \frac{\frac{2701}{6} + 36\zeta(2)}{\varepsilon} + \frac{5945}{8} + \frac{363\zeta(2)}{4} + 12\zeta(3)\right], \tag{30}$$

$$I_{3c} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[-\frac{208}{3\varepsilon^3} - \frac{188}{3\varepsilon^2} - \frac{179 + 26\zeta(2)}{\varepsilon} - \frac{2683}{12} - \frac{47\zeta(2)}{2} + \frac{250\zeta(3)}{3}\right], \tag{31}$$

$$I_{3d} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{12}{\varepsilon^3} + \frac{71}{2\varepsilon^2} + \frac{\frac{1793}{12} + 9\zeta(2)}{2\varepsilon} + \frac{12731}{96} + \frac{213\zeta(2)}{16} - \frac{15\zeta(3)}{2}\right], \tag{32}$$

$$I_{3e} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[-\frac{24}{\varepsilon^3} - \frac{16}{\varepsilon^2} - \frac{34 + 9\zeta(2)}{\varepsilon} - 22 - 6\zeta(2) + 3\zeta(3)\right], \tag{33}$$

$$I_{3f} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{8}{\varepsilon^3} + \frac{28}{3\varepsilon^2} + \frac{5 + 3\zeta(2)}{\varepsilon} - \frac{27}{4} + \frac{7\zeta(2)}{2} + 7\zeta(3)\right], \tag{34}$$

$$I_{3g} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{24}{\varepsilon^3} + \frac{40}{\varepsilon^2} + \frac{50 + 9\zeta(2)}{\varepsilon} + 56 + 15\zeta(2) - 3\zeta(3)\right], \tag{35}$$

$$I_{3h} = 0, \tag{36}$$

$$I_{3i} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{16}{\varepsilon^3} + \frac{92}{3\varepsilon^2} + \frac{35 + 6\zeta(2)}{\varepsilon} + \frac{275}{12} + \frac{23\zeta(2)}{2} - 2\zeta(3)\right], \tag{37}$$

$$I_{3j} = \frac{1}{(4\pi)^6}\left(\frac{e^{\gamma_E}}{4\pi}\right)^{-3\varepsilon/2}\left[\frac{8}{\varepsilon^3} + \frac{8}{\varepsilon^2} + \frac{6 + 3\zeta(2)}{\varepsilon} + 4 + 3\zeta(2) - \zeta(3)\right]. \tag{38}$$

B. Divergent terms through three loops

Denoting by $E_{\rm vac}$ the vacuum energy density in the symmetric phase, we find up to three loops the expansion

$$E_{\rm vac} = \frac{m^4}{g\mu^\varepsilon} Z_v + \hbar E_1 + \hbar^2 E_2 + \hbar^3 E_3 + \mathcal{O}(\hbar^4), \tag{39}$$

with the one-loop part

$$E_1 = m_B^D \frac{N}{2} I_{1a}, \tag{40}$$

the two-loop part

$$E_2 = m_B^{2D-4} \left(\frac{1}{2} e_B^2 \frac{N}{2} I_{2a} - \frac{1}{4} g_B \frac{N(N+2)}{8} I_{2b} \right), \tag{41}$$

and the three-loop part

$$\begin{aligned} E_3 = m_B^{3D-8} \Bigg[& \frac{1}{4} e_B^4 \frac{N}{2} I_{3a} + \frac{1}{2} e_B^4 \frac{N}{2} I_{3b} + \frac{1}{4} e_B^4 \left(\frac{N}{2}\right)^2 I_{3c} - e_B^4 N I_{3d} - \frac{1}{2} e_B^4 \frac{N^2}{2} I_{3e} + \frac{1}{4} e_B^4 2N I_{3f} \\ & - g_B e_B^2 \frac{N(N+2)}{8} I_{3g} + \frac{1}{8} g_B^2 \frac{N(N+2)}{8} I_{3i} + \frac{1}{2} g_B^2 \frac{N(N+2)^2}{32} I_{3j} \Bigg]. \end{aligned} \tag{42}$$

A few remarks are useful on the calculation of the pole terms of the diagrams. Taking into account the expansion of the renormalization constants in the form (5), each Feynman integral is expanded up to order $\hbar^3$. The expansion coefficients are determined to cancel the $1/\varepsilon^n$ terms arising from the Feynman integrals. In particular we have:

– To order $\hbar$: the cancellation of the $1/\varepsilon$ pole leads directly to the known one-loop results (18) and (19), and to (22) as in [15].

– To order $\hbar^2$: there are pole terms of the form $1/\varepsilon^2$, $1/\varepsilon$ and $1/\varepsilon \times \ln(m^2/\bar\mu^2)$. The cancellation of the $1/\varepsilon^2$ and $1/\varepsilon$ poles leads to the renormalization constants (23) and (24). The $1/\varepsilon \times \ln(m^2/\bar\mu^2)$ terms cancel if

$$Z_\phi^{(1)} = Z_{m^2}^{(1)} - \frac{g(N+2) - 12e^2}{2\varepsilon}, \tag{43}$$

which is fulfilled by the one-loop expressions (18) and (20) for $Z_{m^2}^{(1)}$ and $Z_\phi^{(1)}$, respectively. After this, the cancellation of the ordinary poles lets us recover the result (25) obtained before in [15].

– Finally, we need to cancel the poles at the $\hbar^3$ level. Besides poles without logs, there are poles of the types $1/\varepsilon \times \ln$, $1/\varepsilon^2 \times \ln$ and $1/\varepsilon \times \ln^2$, which have to vanish. To simplify the discussion, we introduce the notation $Z^{(i,j)}$ where the first superscript i indicates the loop order, and the second superscript j gives the order of the pole, i.e.: $Z^{(i)} = \sum_{j=1}^{i} Z^{(i,j)}/\varepsilon^j$.

When removing the pole proportional to $1/\varepsilon \times \ln^2$ we obtain $Z_\phi^{(2,2)}$ as a function of $Z_A^{(1,1)}$, $Z_g^{(1,1)}$, $Z_{m^2}^{(1,1)}$ and $Z_{m^2}^{(2,2)}$. Similarly, the removal of $1/\varepsilon \times \ln$ gives $Z_\phi^{(2,1)}$ as a function of $Z_A^{(1,1)}$ and $Z_{m^2}^{(2,1)}$. Finally, the removal of the pole proportional to $1/\varepsilon^2 \times \ln$ gives $Z_g^{(1,1)}$ as a function of $Z_A^{(1)}$ and $Z_{m^2}^{(1,1)}$. Inserting this into the expression for $Z_\phi^{(2,2)}$, we obtain

$$Z_{\phi}^{(2,2)} = e^4(6-5N) + 6e^2 Z_{m^2}^{(1,1)} - \left[\frac{3}{4}g^2(N+2) + \frac{1}{2}g(N+2)Z_{m^2}^{(1,1)} - Z_{m^2}^{(2,2)}\right]. \tag{44}$$

Taking into account all these relations, the $1/\ \epsilon p^n$ pole terms are found removed with $Z_v^{(3,1)}$ and $Z_v^{(3,2)}$ which are function of $Z_A^{(1,1)}$ only, while $Z_v^{(3,3)}$ is determined independently of $Z_A^{(1,1)}$.

Using the known results for $Z_A^{(1,1)}$ and $Z_{m^2}^{(1,1)}$, we recover the result (19) derived before in [15]. Using the known result for $Z_A^{(1,1)}, Z_{m^2}^{(1,1)}, Z_{m^2}^{(2,1)}, Z_{m^2}^{(2,2)}$, we also obtain

$$Z_{\phi}^{(2,1)} = -\frac{1}{16}g^2(N+2) - \frac{1}{12}e^4(11N+18), \tag{45}$$

$$Z_{\phi}^{(2,1)} = e^4(N+18) \tag{46}$$

this coinciding with previous results derived from a renormalization of the two-point functions in Refs. [9,21].

Finally, inserting $Z_A^{(1,1)}$ into $Z_v^{(3,1)}$ and $Z_v^{(3,2)}$, we have

$$Z_v^{(3,1)} = \frac{N(N+2)g^2}{64} - \frac{N[43N + 294 - 384\zeta(3)]e^2}{48}, \tag{47}$$

$$Z_v^{(3,2)} = -\frac{5N(N+2)g^2}{24} + 2N(N+2)ge^2 + \frac{N(25N-38)e^4}{6}, \tag{48}$$

$$Z_v^{(3,3)} = \frac{N(N+2)(N+4)g^2}{8} - 3(N+2)Nge^2, \tag{49}$$

where, as mentioned above, $Z_v^{(3,3)}$ is independent of $Z_A^{(1,1)}$. For $e^2 = 0$ we recover the three-loop result of of the pure ϕ^4 theory of Ref. [20].

VI. RENORMALIZATION GROUP FUNCTION OF THE VACUUM γ_V

The critical behavior of the renormalization constant Z_v is characterized by the finite renormalization group function γ_v defined by the logarithmic derivative [20]

$$\gamma_v = \frac{\mu^{1+\varepsilon}}{m^4}\frac{dE_v^c}{d\mu}. \tag{50}$$

Using standard methods [23], γ_v can be extracted from the simple pole terms of Z_v, whose residue will be denoted by $Z_v^{[1]}$, as [20]:

$$\gamma_v = \left(1 + g\frac{\partial}{\partial g} + \frac{1}{2}e\frac{\partial}{\partial e}\right) Z_v^{[1]}, \tag{51}$$

The results of the last section for the $Z_v^{(l,1)}$ yield

$$\gamma_v = \frac{N}{2}\frac{\hbar}{(4\pi)^2} + 4Ne^2\left[\frac{\hbar}{(4\pi)^2}\right]^2 + \left[\frac{3N(N+2)}{64}g^2 + N\left(-\frac{43}{16}N - \frac{147}{8} + 24\zeta(3)\right)e^4\right]\left[\frac{\hbar}{(4\pi)^2}\right]^3 + \mathcal{O}(\hbar^4). \tag{52}$$

VII. VACUUM ENERGY DENSITY

In the previous section, we have focused on the removal of divergences, thus fixing the renormalization constants. Since the three-loop integrals I_{3a}—I_{3j} are known to zeroth order in ε^0, we can also determine the finite vacuum energy density of the symmetric phase of the Ginzburg-Landau model. Up to a negative sign, it is given by the sum of vacuum diagrams. Having in mind the application of our result to phase transitions in three dimensions, we give here its ε expansion. We must calculate the one-loop diagram up to the order ε^2 and the two-loop diagrams up to the order ε.

The general form of the L-loop result has ε expansion of the form

$$E_{\rm vac} = \frac{Nm^4}{4\mu^\varepsilon} \sum_{l=1}^{L} \left[\frac{\hbar}{(4\pi)^2}\right]^l \sum_{k=0}^{L-l} \varepsilon^k E_{lk}, \tag{53}$$

where we have assumed that e^2 and g are of order ε, which is correct at the fixed point relevant for the neighborhood of the phase transition. The expansion coefficients are

$$E_{10} = \ln\left(\frac{m^2}{\bar{\mu}^2}\right) - \frac{3}{2}, \tag{54}$$

$$E_{11} = -\frac{1}{4}\ln^2\left(\frac{m^2}{\bar{\mu}^2}\right) + \frac{3}{4}\ln\left(\frac{m^2}{\bar{\mu}^2}\right) - \frac{7}{8} - \frac{\zeta(2)}{4}, \tag{55}$$

$$E_{12} = \frac{1}{24}\ln^3\left(\frac{m^2}{\bar{\mu}^2}\right) - \frac{3}{16}\ln^2\left(\frac{m^2}{\bar{\mu}^2}\right) + \left[\frac{7}{16} + \frac{\zeta(2)}{8}\right]\ln\left(\frac{m^2}{\bar{\mu}^2}\right) - \frac{15}{32} - \frac{3\zeta(2)}{16} + \frac{\zeta(3)}{12}, \tag{56}$$

$$E_{20} = \left(\frac{N+2}{4}g - 3e^2\right)\ln^2\left(\frac{m^2}{\bar{\mu}^2}\right) - \left(\frac{N+2}{2}g - 14e^2\right)\ln\left(\frac{m^2}{\bar{\mu}^2}\right) + \frac{N+2}{4}g - 19e^2, \tag{57}$$

$$\begin{aligned} E_{21} = &\left(-\frac{N+2}{8}g + \frac{3}{2}e^2\right)\ln^3\left(\frac{m^2}{\bar{\mu}^2}\right) + \left[\frac{3(N+2)}{8}g - \frac{17}{2}e^2\right]\ln^2\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \left[-\frac{(4+\zeta(2))(N+2)}{8}g + \frac{44+3\zeta(2)}{2}e^2\right]\ln\left(\frac{m^2}{\bar{\mu}^2}\right) + \frac{[2+\zeta(2)](N+2)}{8}g - \frac{50+7\zeta(2)}{2}e^2, \end{aligned} \tag{58}$$

$$\begin{aligned} E_{30} = &\left[\frac{(N+2)(N+4)}{16}g^2 - \frac{3(N+2)}{2}ge^2 + \frac{5N+48}{6}e^4\right]\ln^3\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \left[-\frac{(N+2)(2N+15)}{16}g^2 + 7(N+2)ge^2 + \frac{49N-438}{12}e^4\right]\ln^2\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \left[\frac{(N+2)(2N+39)}{32}g^2 - \frac{23(N+2)}{2}ge^2 + \frac{-139N+290+384\zeta(3)}{8}e^4\right]\ln\left(\frac{m^2}{\bar{\mu}^2}\right) \\ &+ \frac{N+2}{192}g^2 + 6(N+2)ge^2 \\ &+ \left[\frac{1351N}{48} + \frac{261}{8} - \left(\frac{56N}{3} + 208\right)\zeta(3) + 176\zeta(4) + 64\zeta(2)\ln^2 2 - \frac{32}{3}\ln^4 2 - 256\,\mathrm{Li}_4\left(\frac{1}{2}\right)\right]e^4. \end{aligned} \tag{59}$$

VIII. CONCLUSION

With the help of dimensional regularization and in the modified minimal subtraction scheme $\overline{\mathrm{MS}}$ we have computed the vacuum energy density in an ε expansion up to three loops for the symmetric phase of the Ginzburg-Landau model. We have further determined the renormalization group function of the vacuum γ_v, which is relevant for calculating amplitude ratios and is, in the present scheme, same above and below T_c. To arrive at the final goal of deriving universal quantities amplitude ratios for the specific heat above and below the phase transition, we must do a similar calculation also in the ordered phase below T_c. There calculations will be complicated by a proliferation of Feynman diagrams, the appearance of more mass scales and infrared divergences for $N > 2$.

Fortunately, the amplitude ratio of the specific heat does not require knowledge of the full effective potential, but only its value at the minimum, whose evaluation is simpler and will be given in future work.

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[1] B. I. Halperin, T. C. Lubensky and S.-K. Ma, Phys. Rev. Lett. **32**, 292 (1974); J.-H. Chen, T. C. Lubensky and D. R. Nelson, Phys. Rev. B **17**, 4274 (1978)

[2] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1983). For a comparison of the two theories see H. Kleinert, Phys. Lett. B **128**, 69 (1983) (http://www.physik.fu-berlin.de/~kleinert/106).

[3] V.L. Ginzburg, Fiz. Twerd. Tela **2**, 2031 (1960) [Sov. Phys. Solid State **2**, 1824 (1961)]. See also the detailed discussion in Chapter 13 of the textbook L.D. Landau and E.M. Lifshitz, *Statistical Physics*, 3rd edition, Pergamon Press, London, 1968; and by C.J. Lobb, Phys. Rev. B **36**, 3930 (1987).

[4] *Criterion for Dominance of Directional versus Size Fluctuations ¡br¿of Order Field in Restoring Spontaneously Broken Continuous Symmetries* , Phys. Rev. Lett. **84**, 286 (2000)

[5] J. Bartholomew, Phys. Rev. B **28**, 5378 (1983); Y. Munehisa, Phys. Lett. B **155**, 159 (1985).

[6] J. Als-Nielsen et al., Phys. Rev. B **22**, 312 (1980).

[7] H. Kleinert, Lett. Nuovo Cimento **35**, 405 (1982) (http://www.physik.fu-berlin.de/~kleinert/97).

[8] For details of the derivation see Chapter 13 in the textbook by H. Kleinert, *Gauge Fields in Condensed Matter, vol. 1*, (World Scientific, Singapore, 1989), readable in the internet at http://www.physik.fu-berlin.de/~kleinert/re.html#b1.

[9] J. Tessmann, MS thesis 1984 written under the supervision of one the present authors (HK).

[10] R. Folk and Y. Holovatch, J. Phys. A **29**, 3409 (1996).

[11] C. de Calan and F. S. Nogueira, Phys. Rev. B **60**, 4255 (1999).

[12] F. S. Nogueira, Phys. Rev. B **62**, 14559 (2000).

[13] H. Kleinert and F. S. Nogueira, *Charged Fixed Point Found in Superconductor Below T_c and New Expansion Around $\kappa = 1/\sqrt{2}$*, cond-mat/0104573.

[14] For the definition of various subtraction schemes see Chapter 13 in the textbook H. Kleinert and V. Schulte-Frohlinde, *Critical Phenomena in Φ^4-Theory*, World Scientific., Singapore 2001, pp. 1–487 (http://www.physik.fu-berlin.de/~kleinert/b8)

[15] H. Kleinert and B. Van den Bossche, *Two-loop effective potential of $O(N)$-symmetric scalar QED in $4-\varepsilon$ dimensions*, cond-mat/0104102

[16] H. Kleinert, A. Pelster and B. Van den Bossche, *Recursive graphical construction for Feynman diagrams and their weights in Ginzburg-Landau theory*, hep-th/0107017 in preparation

[17] D. J. Broadhurst, Z. Phys. C **54**, 599 (1992);
D. J. Broadhurst, Eur. Phys. J. C **8**, 311 (1999);
J.-M. Chung and B. K. Chung, J.Korean Phys.Soc. **38**, 60 (2001)

[18] F. S. Nogueira, Europhys. Lett. **45**, 612 (1999).

[19] C. de Calan and F. S. Nogueira, Phys. Rev. B **60**, 11929 (1999).

[20] B. Kastening, Phys. Rev. D **54**, 3965 (1996);
B. Kastening, Phys. Rev. D **57**, 3567 (1998)

[21] S. Kolnberger, R. Folk, Phys. Rev. B **41**, 4083 (1990);

R. Folk, Yu. Holovatch, *Critical fluctuations in normal-to-superconducting transition*, cond-mat/9807421

[22] For details see Section 8.3 in the textbook cited in [14].

[23] For details see Section 10.3 in the textbook cited in [14].

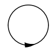
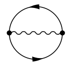
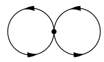
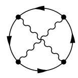
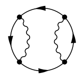
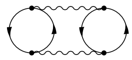
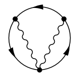
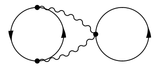
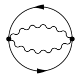
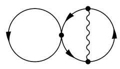
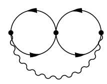
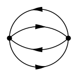
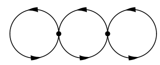
L, n_1, n_2, n_3	
$1, 0, 0, 0$	1 
$2, 0, 0, 2$	$\frac{1}{2}$ 
$2, 1, 0, 0$	$\frac{1}{2}$ 
$3, 0, 0, 4$	$\frac{1}{4}$  $\frac{1}{2}$  $\frac{1}{4}$ 
$3, 0, 1, 2$	1  $\frac{1}{2}$ 
$3, 0, 2, 0$	$\frac{1}{4}$ 
$3, 1, 0, 2$	1  $\frac{1}{2}$ 
$3, 2, 0, 0$	$\frac{1}{8}$  $\frac{1}{2}$ 

TABLE I. 1PI vacuum diagrams $W^{(L,n_1,n_2,n_3)}$ and their weights through the three-loop order of the $O(N)$ Ginzburg–Landau model, where L denotes the loop order and n_1, n_2, n_3 count the number of g, e^2 and e vertices, respectively.