Stochastic Calculus for Assets with Non-Gaussian Price Fluctuations *

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From the path integral formalism for price fluctuations with non-Gaussian distributions I derive a simple new stochastic calculus to replace Itô's calculus for stochastic fluctuations.

I. INTRODUCTION

Following Pareto's observation in the 19th century [1], Mandelbrot emphasized in the 1960s that the logarithms of assets prices in financial markets do not fluctuate with Gaussian distributions, but possess much larger tails which may be approximated by various other distributions, most prominently the truncated Lévy distributions [2–4]. This has the consequence that the associated stochastic differential equations cannot be treated with the popular Itô calculus. The purpose of this paper is to develop the appropriate calculus to replace it. In Section II we briefly recapitulate the Gaussian approximation and set the stage for the generalization in Section III.

II. GAUSSIAN APPROXIMATION TO FLUCTUATION PROPERTIES OF STOCK PRICES

Let $S(t)$ denote the price of some stock. Over long time spans, the average over many stock prices has a time behavior which can be approximated by pieces of exponentials. This is why they are usually plotted on a logarithmic scale. For an illustration see the Dow-Jones industrial index over 60 years in Fig. 1.
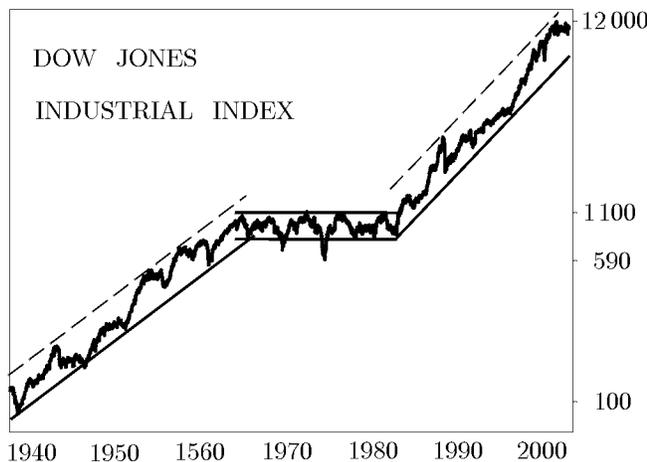

FIG. 1. Periods of exponential growth of average over major industrial stocks in the Unites States over 60 years.

For a liquid market with many particiapants, the price fluctuations seem to be driven by a stochastic noise with a white spectrum, as illustrated on Fig. 2.
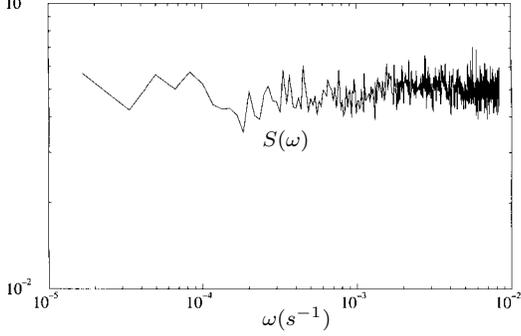

FIG. 2. Frequency-independence of spectrum of the fluctuations of the exchange rate DM/US$ as a function of the frequency, indicating a white noise in the associated stochastic differential equation (1) (from the textbook [5]).

Over longer time spans, the volatility changes stochastically, as illustrated by the data of the S&P500 index over the years 1984-1997 shown in Fig. 3. In particular, there are strong increases short before market crashes.
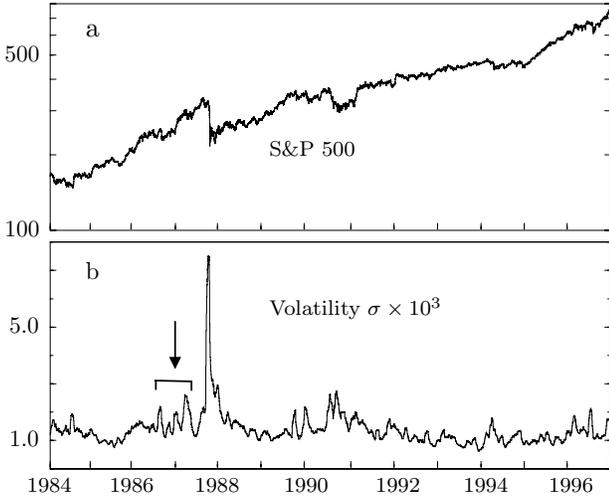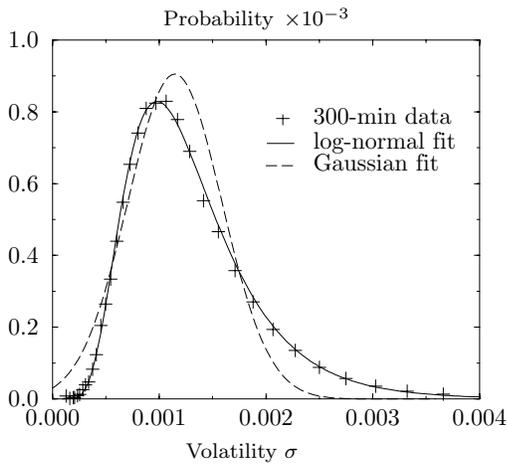

FIG. 3. (a) The S&P 500 index for the 13-year period 1 Jan 1984 - 31 Dec 1996 at interval of 1 min. Large fluctuations appeared on 19 Oct 1987 (*black Monday*). (b) Volatility with T=1 mon (8190 min) and time lag 30min. The precursors of the '87 crash are indicated by arrows (from Ref. [6]).

The distribution of the logarithms is Fig. 4. The theory to be developed here will ignore these fluctuations and assume a constamt volatility. For recent work taking them into account see [7].

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FIG. 4. Comparison of the best log-normal and Gaussian fits for the 300-min data (from Ref. [6]).

An individual stock will in general have larger volatility than an averaged market index, especially when the company small and the number of traded stocks per day is small.

In lowest approximation, the stockprice $S(t)$ satisfies the simplest stochastic differential equation for exponential growth

$$\frac{\dot{S}(t)}{S(t)} = r_S + \eta(t), \tag{1}$$

where where r_S is the average slope in the logarithmic plot of the type (1), and $\eta(t)$ is a white noise of unit strength with the correlation functions

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \sigma^2 \delta(t - t'). \tag{2}$$

The standard deviation σ enters into the volatility of the stock price, which is measured by the expectation value

$$\left\langle \left[\frac{\dot{S}(t)}{S(t)} \right]^2 \right\rangle = \sigma^2. \tag{3}$$

The logarithm of the stockprice

$$x(t) = \log S(t) \tag{4}$$

does not simply satisfy the differential equation $x(t) = \dot{S}(t)/S(t) = r_S + \sigma\eta(t)$. There is a correction due the stochastic nature of $x(t)$ and $S(t)$. Recall that according to Itô's rule we may expand

$$\dot{x}(t) = \frac{dx}{dS}\dot{S}(t) + \frac{1}{2}\frac{d^2x}{dS^2}\dot{S}^2(t)\,dt + \ldots$$

$$= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\left[\frac{\dot{S}(t)}{S(t)}\right]^2 dt + \ldots, \tag{5}$$

and replace the last term by its expectation value (3):

$$\dot{x}(t) = \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\left\langle \left[\frac{\dot{S}(t)}{S(t)}\right]^2 \right\rangle dt$$

$$= \frac{\dot{S}(t)}{S(t)} - \frac{1}{2}\sigma^2 + \ldots . \tag{6}$$

Inserting here Eq. (1), we obtain

$$\dot{x}(t) = r_x + \eta(t), \tag{7}$$

where linear growth rate r_x is related to the exponential growth rate r_S in (1) by

$$r_x = r_S - \frac{1}{2}\sigma^2. \tag{8}$$

In praxis, this relation implies that if we fit a straight line through a plot of the logaritms of stock prices, the forward extrapolation of the average stock price is given by

$$\langle S(t) \rangle = S(0)\,e^{r_S t} = S(0)\,e^{(r_x + \sigma^2/2)t}. \tag{9}$$

A typical set of solutions of the stochasic differential equation (7) is shown in Fig. 5.
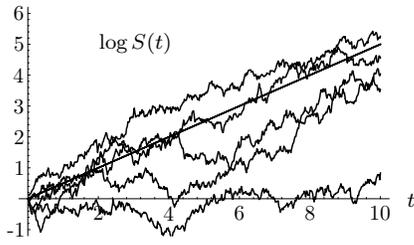
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FIG. 5. Behavior of logarithm of stock price following the stochastic differential equation (1).

III. LÉVY DISTRIBUTIONS

The description of the fluctuations of the logarithms of the stock prices around the linear trend by a Gaussian distribution of a fixed width is only a rough approximation to the real stock prices. As explained before, these have volatilities depending on time. More severely, they have distributions in which rare events have a much higher relative probability than in Gaussian distributions. They can be fitted much better with the help of Lévy distributions [2]. These distributions are defined by the Fourier transform

$$\tilde{L}^\mu_{\sigma^2}(x) \equiv \int_\infty^\infty \frac{dp}{2\pi} L^\mu_{\sigma^2}(p)\, e^{ipx}, \tag{10}$$

with

$$L^\mu_{\sigma^2}(p) \equiv \exp\left[-(\sigma^2 p^2)^{\mu/2}/2\right]. \tag{11}$$

It reduces to the Gaussian distribution for $\mu = 2$, where the exponential reduces to an ordinary kinetic energy with a mass $M = 1/\sigma^2$. The characteristic property of the Lévy distributions is that they behave for large x like a power of x:

$$\tilde{L}^\mu_{\sigma^2}(x) \to A^\mu_{\sigma^2} \frac{\mu}{|x|^{1+\mu}}. \tag{12}$$

These power falloffs are refered to as *paretian tails* of the distributions. The amplitude of the tails is found by approximating the integral (10) for large x, where only small momenta contribute, by

$$\tilde{L}^\mu_{\sigma^2}(x) \approx \int_{-\infty}^\infty \frac{dp}{2\pi} \left[1 - \frac{1}{2}(\sigma^2 p^2)^{\mu/2}\right] e^{ipx} \underset{x\to\infty}{\to} A^\mu_{\sigma^2} \frac{\mu}{|x|^{1+\mu}}, \tag{13}$$

with

$$A^\mu_{\sigma^2} = -\frac{\sigma^\mu}{2\mu} \int_0^\infty \frac{dp'}{\pi} p'^\mu \cos p' = \frac{\sigma^\mu}{2\pi\mu} \sin(\pi\mu/2)\, \Gamma(1+\mu). \tag{14}$$

The stock market data are fitted best with μ between 1.2 and 1.5, and we shall use $\mu = 3/2$ most of the time for simplicity, where one has

$$A^{3/2}_{\sigma^2} = \frac{1}{4} \frac{\sigma^{3/2}}{\sqrt{2\pi}}. \tag{15}$$

The full Taylor expansion of the Fourier transform (11) yields the asymptotic expansion

$$\tilde{L}^\mu_{\sigma^2}(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^\infty \frac{dp}{\pi} \frac{\sigma^{\mu n} p^{\mu n}}{2^n} \cos px = \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n!} \frac{\sigma^{\mu n}}{2^n \pi} \Gamma(1+n\mu) \sin(\pi\mu/2) \frac{1}{|x|^{1+\mu}}. \tag{16}$$

This expansion is not useful for practical calculations since being a large-x expansion, it fails to reproduce the exponential tails of the distribution. In particular, it does not reduce to the Gaussian distribution in the limit $\mu \to 2$.

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A. Truncated Lévy Distributions

An undesirable property of the Lévy distributions which is incompatible with financial data is that they have an infinite fluctuation width for $\mu < 2$, since

$$\sigma^2 = \langle x^2 \rangle \equiv \int_{-\infty}^{\infty} dx\, x^2\, \tilde{L}^{\mu}_{\sigma^2}(x) = -\frac{d^2}{dp^2} L^{\mu}_{\sigma^2}(p)\bigg|_{p=0}. \tag{17}$$

In contrast, real stock prices do have a finite width. To account for both parentian tails *and* finite width one introduces the so-called *truncated Lévy distributions*. They are defined by

$$\tilde{L}^{(\mu,\alpha)}_{\sigma^2}(x) \equiv \int_{\infty}^{\infty} \frac{dp}{2\pi} L^{(\mu,\alpha)}_{\sigma^2}(p)\, e^{ipx}, \tag{18}$$

with a Fourier transform $L^{(\mu,\alpha)}_{\sigma^2}(p)$ which generalizes the function (11). It will be convenient to write it as an exponential [5]

$$L^{(\mu,\alpha)}_{\sigma^2}(p) \equiv e^{-H(p)}, \tag{19}$$

with a "Hamiltonian function"

$$\begin{aligned}
H(p) &\equiv \frac{\sigma^2}{2}\frac{\alpha^{2-\mu}}{\mu(1-\mu)}\left[(\alpha+ip)^{\mu}+(\alpha-ip)^{\mu}-2\alpha^{\mu}\right] \\
&= \sigma^2 \frac{(\alpha^2+p^2)^{\mu/2}\cos[\mu\arctan(p/\alpha)]-\alpha^{\mu}}{\alpha^{\mu-2}\mu(1-\mu)}.
\end{aligned} \tag{20}$$

The asymptotic behavior of the truncated Lévy distributions differs from the power behavior of the Lévy distribution in Eq. (13) by an exponential factor $e^{-\alpha x}$ which guarantees the finiteness of the width σ and of all higher moments. The leading term is again obtained from the Fourier transform

$$\begin{aligned}
\tilde{L}^{(\mu,\alpha)}_{\sigma^2}(x) &\approx \int_{-\infty}^{\infty} \frac{dp}{2\pi}\left\{1-\frac{\sigma^2}{2}\frac{\alpha^{2-\mu}}{\mu(1-\mu)}\left[(\alpha+ip)^{\mu}+(\alpha-ip)^{\mu}-2\alpha^{\mu}\right]\right\}e^{ipx} \\
&\xrightarrow[x\to\infty]{} \sigma^2\Gamma(1+\mu)\frac{\sin(\pi\mu)}{2\pi}\frac{\alpha^{2-\mu}}{\mu(1-\mu)}\frac{e^{-\alpha|x|}}{|x|^{1+\mu}}.
\end{aligned} \tag{21}$$

This follows directly from the integral formulas [8]

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}(\alpha+ip)^{\mu}e^{ipx}=\frac{\Theta(x)}{\Gamma(-\mu)}\frac{e^{-\alpha x}}{x^{1+\mu}}, \qquad \int_{-\infty}^{\infty}\frac{dp}{2\pi}(\alpha-ip)^{\mu}e^{ipx}=\frac{\Theta(-x)}{\Gamma(-\mu)}\frac{e^{-\alpha|x|}}{|x|^{1+\mu}}, \tag{22}$$

and the identity for Gamma functions $1/\Gamma(-z)=-\Gamma(1+z)\sin(\pi z)/\pi$. The full expansion is integrated with the help of the formula [9]

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{dp}{2\pi}(\alpha+ip)^{\mu}(\alpha-ip)^{\nu}e^{ipx} \\
&= (2\alpha)^{\mu/2+\nu/2}\frac{1}{|x|^{1+\mu/2+\nu/2}}\begin{cases}\dfrac{1}{\Gamma(-\mu)}W_{(\nu-\mu)/2,(1+\mu+\nu)/2}(2\alpha x) & x>0, \\[2mm] \dfrac{1}{\Gamma(-\nu)}W_{(\mu-\nu)/2,(1+\mu+\nu)/2}(2\alpha x) & x<0,\end{cases}\quad\text{for} \tag{23}
\end{aligned}$$

where the Whittaker functions $W_{(\nu-\mu)/2,(1+\mu+\nu)/2}(2\alpha x)$ can be expressed in terms of Kummer's confluent hypergeometric function $_1F_1(a;b;z)$ as

$$\begin{aligned}
W_{\lambda,\kappa}(z) &= \frac{\Gamma(-2\kappa)}{\Gamma(1/2-\kappa-\lambda)}z^{\kappa+1/2}e^{-z/2}{}_1F_1(1/2+\kappa-\lambda;2\kappa+1;z) \\
&+ \frac{\Gamma(2\kappa)}{\Gamma(1/2+\kappa-\lambda)}z^{-\kappa+1/2}e^{-z/2}{}_1F_1(1/2-\kappa-\lambda;-2\kappa+1;z).
\end{aligned} \tag{24}$$

For $\nu = 0$, only $x > 0$ gives a nonzero integral (23), which reduces with $W_{-\mu/2, 1/2+\mu/2}(z) = z^{-\mu/2}e^{-z/2}$ properly to the left equation in (22). Setting $\mu = \nu$ we find

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha^2 + p^2)^\nu e^{ipx} = (2\alpha)^{\nu/2} \frac{1}{|x|^{1+\nu}} \frac{1}{\Gamma(-\nu)} W_{0,1/2+\nu}(2\alpha|x|). \tag{25}$$

Inserting

$$W_{0,1/2+\nu}(z) = \sqrt{\frac{2z}{\pi}} K_{1/2+\nu}(z/2), \tag{26}$$

this becomes

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} (\alpha^2 + p^2)^\nu e^{ipx} = \left(\frac{2\alpha}{|x|}\right)^{1/2+\nu} \frac{1}{\sqrt{\pi}\Gamma(-\nu)} K_{1/2+\nu}(\alpha|x|). \tag{27}$$

For $\nu = -1$ where $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\pi/2z}e^{-z}$, this reduces to

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{\alpha^2 + p^2} e^{ipx} = \frac{1}{2\alpha} e^{-\alpha|x|}. \tag{28}$$

In contrast to a Gaussian distribution, which are characterized completely by the width σ, the truncated Lévy distributions contain three parameters σ^2, μ, and α. Best fits to two types of fluctuating market prices are shown in Fig. 6, in which we plot the cumulative probabilities

$$P_<(\delta x) = \int_{-\infty}^{\delta x} dx\, \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x), \qquad P_>(\delta x) = \int_{\delta x}^{\infty} dx\, \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x) = 1 - P_<(\delta x). \tag{29}$$

For negative price fluctuations δx, the plot shows $P_<(\delta x)$, for positive price fluctuations $P_>(\delta x)$. By definition, $P_<(-\infty) = 0$, $P_<(0) = 1/2$, $P_<(\infty) = 1$ and $P_>(-\infty) = 1$, $P_>(0) = 1/2$, $P_>(\infty) = 0$. To fit the general shape, one chooses an appropriate parameter μ which turns out to be rather universal, close to $\mu = 3/2$. The remaining two parameters fix all expansion coefficients of Hamiltonian (20):

$$H(p) = \frac{1}{2}c_2\, p^2 - \frac{1}{4!}c_4\, p^4 + \frac{1}{6!}c_6\, p^6 - \frac{1}{8!}c_8\, p^8 + \dots . \tag{30}$$

The numbers c_n are refered to as the *cumulants* of the truncated Lévy distribution. They are equal to

$$
\begin{aligned}
c_2 &= \sigma^2, \\
c_4 &= \sigma^2(2-\mu)(3-\mu)\alpha^{-2}, \\
c_6 &= \sigma^2(2-\mu)(3-\mu)(4-\mu)(5-\mu)\alpha^{-4}, \\
&\vdots \\
c_{2n} &= \sigma^2 \frac{\Gamma(2n-\mu)}{\Gamma(2-\mu)} \alpha^{2-2n},
\end{aligned}
\tag{31}
$$

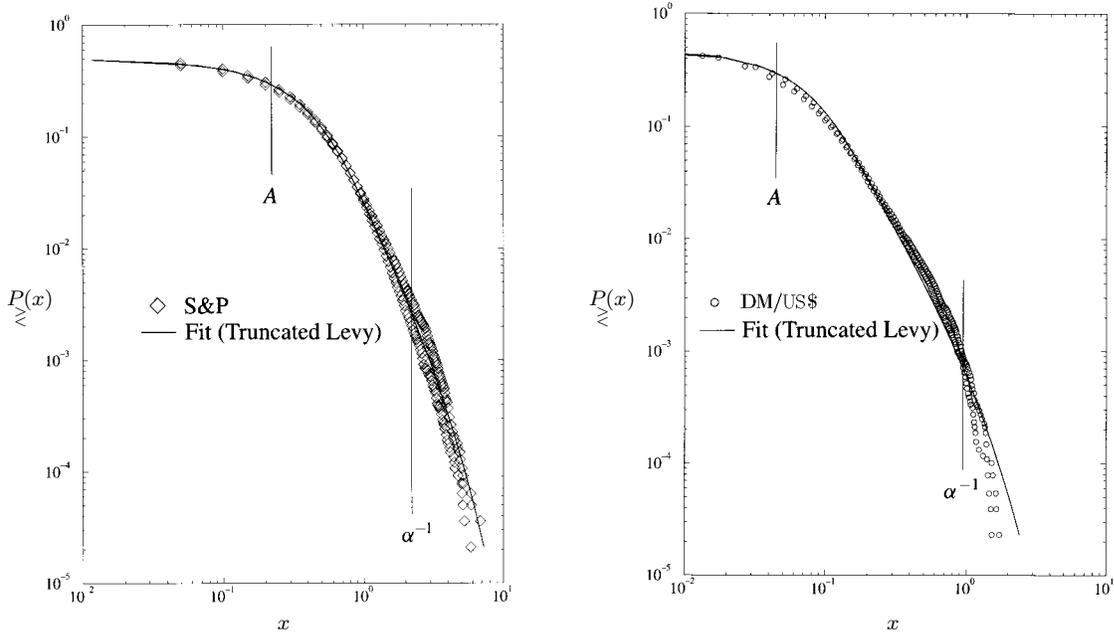FIG. 6. Best fit of cumulative versions (29) of truncated Lévy distribution, $P_<(\delta x)$ for negative price fluctuations δx, and $P_>(\delta x)$ for positive price fluctuations to the instantaneous fluctuations of the S&P 500 (l.h.s) and the DM/US\$ exchange rate measured every fifteen minutes. The negative fluctuations lie on a slightly higher curve than the positive ones. The difference will be neglected. The parameters A and α are the size and truncation parameters of the distribution. The best parameter μ is $3/2$ (from [5]).

The first determines the quadratic fluctuation width

$$\langle x^2 \rangle \equiv \int_{-\infty}^{\infty} dx\, x^2\, \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x) = \left. -\frac{d^2}{dp^2} L_{\sigma^2}^{(\mu,\alpha)}(p) \right|_{p=0} = c_2 = \sigma^2, \tag{32}$$

the second the fourth-order expectation

$$\langle x^4 \rangle \equiv \int_{-\infty}^{\infty} dx\, x^4\, \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x) = \left. \frac{d^4}{dp^4} L_{\sigma^2}^{(\mu,\alpha)}(p) \right|_{p=0} = c_4 + 3c_2^2, \tag{33}$$

and so on:

$$\langle x^6 \rangle = c_6 + 15c_4 c_2 + 15c_2^3, \qquad \langle x^8 \rangle = c_8 + 28c_6 c_2 + 35c_4^2 + 210c_4 c_2^2 + 105c_2^4, \dots \ . \tag{34}$$

In analyzing the data, one usually defines the so-called *kurtosis*, which is the normalized fourth-order cumulant

$$\kappa \equiv \bar{c}_4 \equiv \frac{c_4}{c_2^2} = \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 3. \tag{35}$$

It depends on the parameters σ^2, μ, α as follows

$$\kappa = \frac{(2-\mu)(3-\mu)}{\sigma^2 \alpha^2}. \tag{36}$$

Given the volatility σ and the kurtosis κ, we extract the Lévy parameter α from the equation

$$\alpha = \frac{1}{\sigma} \sqrt{\frac{(2-\mu)(3-\mu)}{\kappa}}. \tag{37}$$

In terms of κ and σ^2, the expansion coefficients are

$$\bar{c}_4 = \kappa, \quad \bar{c}_6 = \kappa^2 \frac{(5-\mu)(4-\mu)}{(3-\mu)(2-\mu)}, \quad \bar{c}_8 = \kappa^2 \frac{(7-\mu)(6-\mu)(5-\mu)(4-\mu)}{(3-\mu)^2(2-\mu)^2},$$

$$\vdots$$

$$\bar{c}_n = \kappa^{n/2-1} \frac{\Gamma(n-\mu)/\Gamma(4-\mu)}{(3-\mu)^{n/2-2}(2-\mu)^{n/2-2}}. \tag{38}$$

For $\mu = 3/2$, the second equation in (37) becomes simply

$$\alpha = \frac{1}{2}\sqrt{\frac{3}{\sigma^2 \kappa}}, \tag{39}$$

and the coefficients (40):

$$\bar{c}_4 = \kappa, \quad \bar{c}_6 = \frac{5 \cdot 7}{3} \kappa^2, \quad \bar{c}_8 = 5 \cdot 7 \cdot 11 \, \kappa^2,$$

$$\vdots$$

$$\bar{c}_n = \frac{\Gamma(n-3/2)/\Gamma(5/2)}{3^{n/2-2}/2^{n-4}} \kappa^{n/2-1}. \tag{40}$$

For zero kurtosis, the truncated Lévy distribution reduces to a Gaussian distribution of width σ. For a study of the approach see Refs. [3,4]. The change in shape for a fixed width and increasing kurtosis is shown in Fig. 7.
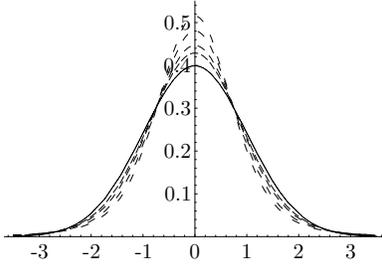

FIG. 7. Change in shape of truncated Lévy distributions of width $\sigma = 1$ with increasing kurtoses $\kappa = 0$ (Gaussian, solid curve), 1, 2, 5, 10.

From the S&P and DM/US\$ data with time intervals $\Delta t = 15\,\mathrm{min}$ one extracts $\sigma^2 = 0.280$ and 0.0163, and the kurtoses $\kappa = 12.7$ and 20.5, respectively. This implies $M \approx 3.57$ and $M \approx 61.35$, and $\alpha \approx 0.46$ and $\alpha \approx 1.50$, respectively.

The other normalized cumulants $(\bar{c}_6, \bar{c}_8)$ are then all determined to be $(1881.72, 788627.46, 6.51012 \times 10^8)$ and $(-4902.92, 3.3168 \times 10^6 , 4.4197 \times 10^9)$, repectively. The cumulants increase rapidly showing that the expansion needs resummation.

From the data, the other normalized cumulants are found by evaluating the ratios of expectation values

$$\bar{c}_6 = \frac{\langle x^6 \rangle}{\langle x^2 \rangle^3} - 15 \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} + 30,$$

$$\bar{c}_8 = \frac{\langle x^8 \rangle}{\langle x^2 \rangle^4} - 28 \frac{\langle x^6 \rangle}{\langle x^2 \rangle^3} - 35 \frac{\langle x^4 \rangle^2}{\langle x^2 \rangle^4} + 420 \frac{\langle x^4 \rangle}{\langle x^2 \rangle^2} - 630, \ldots . \tag{41}$$

Note that for a truncated Lévy distribution, the expectation value of an exponential is given by

$$\langle e^{Px} \rangle \equiv \int dx \, \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x) e^{Px} = \int dx \int \frac{dp}{2\pi} e^{-H(p)} e^{ipx+Px} = e^{-H(-iP)}. \tag{42}$$

B. Path Integral for Truncated Lévy Distribution

It is easy to calculate the properties of the simplest process whose fluctuations are distributed according to a truncated Lévy distribution. The associated stochastic differential equation reads

$$\dot{x}(t) = \eta(t), \tag{43}$$

and the probability distribution of the endpoints of paths starting at a certain initial point is given by the path integral

$$(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right] \delta[\dot{x} - \eta], \tag{44}$$

with the initial condition $x(t_a) = x_a$. The final point is, of course, $x_b = x(t_b)$. The function $\tilde{H}(\eta)$ is defined by the negative logarithm of the truncated Lévy distribution, such that

$$e^{-\tilde{H}(x)} \equiv \tilde{L}_{\sigma^2}^{(\mu,\alpha)}(x). \tag{45}$$

The lowest two correlation functions of the noise in the path integral (44) are given by a straightforward functional generalization of formulas (32)–(34):

$$\langle \eta(t_1)\eta(t_2)\rangle \equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1)\eta(t_2) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]$$
$$= c_2 \delta(t_1 - t_2) = \sigma^2 \delta(t_1 - t_2), \tag{46}$$

$$\langle \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4)\rangle \equiv Z^{-1} \int \mathcal{D}\eta\, \eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]$$
$$= c_4 \delta(t_1 - t_2)\delta(t_1 - t_3)\delta(t_1 - t_4)$$
$$+ c_2^3 \left[\delta(t_1 - t_2)\delta(t_3 - t_4) + \delta(t_1 - t_3)\delta(t_2 - t_4) + \delta(t_1 - t_4)\delta(t_2 - t_3)\right], \tag{47}$$

where Z is the normalization integral

$$Z \equiv \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\, \tilde{H}(\eta(t))\right]. \tag{48}$$

The higher correlation functions are obvious generalizations of (34). All odd correlation functions vanish trivially.

The two different contributions on the right-hand side of Eq. (33) are now distinguishable by their connectedness structure.

An important property of the probability (44) is that it satisfies the semigroup property path integrals

$$(x_c t_c | x_a t_a) = \int dx_b\, (x_c t_c | x_b t_b)(x_b t_b | x_a t_a). \tag{49}$$

Let us check that the experimental asset distributions satisfy this property. This is shown in Fig. 8. Apart from the far ends of the tails, the semigroup property (49) is reasonably well satisfied (from Ref. [5]). The discrepancy manifests itself also at another place: After Eq. (53) we shall see that the solution of the path integral has a kurtosis decreasing inversely proportional to the time. The data in Fig. 8, however, show only an inverse square-root falloff. This can be accounted for in the theory by including fluctuations of the width σ, which are certainly present as was illustrated before in Figs. 3 and 4. For calculations of this type with Gaussian distributions see Ref. [7]. If the semigroup property was satisfied perfectly, the Lévy parameter α would be time independent as we can see from Eq. (37) with $\sigma^2 \propto (t_b - t_a)$ and $\kappa \propto 1/(t_b - t_a)$. With the slower falloff of $\kappa \propto 1/\sqrt{(t_b - t_a)}$, however, α decreases like $1/\sqrt{(t_b - t_a)}$. This is, incidentally, a severe obstacle to finding a Black-Scholes type of price formula for options.
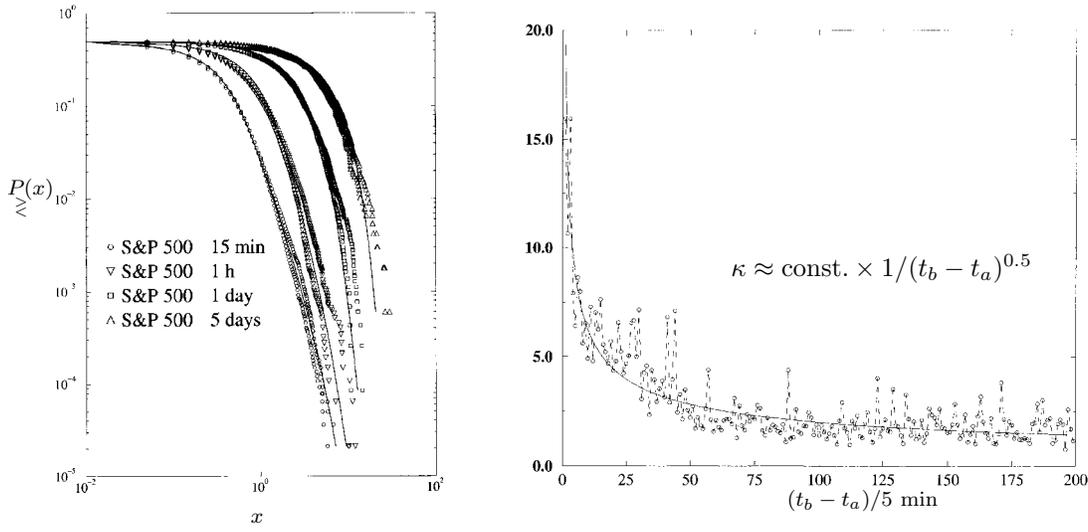
FIG. 8. Left-hand side: Cumulative distributions obtained from repeated convolutions of the 15-min distribution. Apart from the tails, the semigroup property (49) is reasonably well satisfied. Right-hand side: Falloff of kurtosis is slower than $1/(t_b - t_a)$ expected from convolution property [10].

C. Fokker-Planck-Type Equation

The δ-functional may be represented by a Fourier integral leading to

$$(x_b t_b | x_a t_a) = \int \mathcal{D}\eta \int \frac{\mathcal{D}p}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ip(t)\dot{x}(t) - ip(t)\eta(t) - \tilde{H}(\eta(t)) \right] \right\}. \tag{50}$$

Integrating out the noise variable $\eta(t)$ amounts to performing the inverse Fourier tranform (18) at each instant of time and we obtain

$$(x_b t_b | x_a t_a) = \int \frac{\mathcal{D}p}{2\pi} \exp \left\{ \int_{t_a}^{t_b} dt \left[ip(t)\dot{x}(t) - H(p(t)) \right] \right\}. \tag{51}$$

Integrating over all $x(t)$ with fixed end points enforces a constant momentum along the path, and we remain with a single integral

$$(x_b t_b | x_a t_a) = \int \frac{dp}{2\pi} \exp \left[-(t_b - t_a)H(p) + ip(x_b - x_a) \right]. \tag{52}$$

The Fourier integral can now be performed and we obtain

$$(x_b t_b | x_a t_a) = \tilde{L}_{\sigma^2(t_b - t_a)}^{(\mu,\alpha)}(x_b - x_a). \tag{53}$$

The result is therefore a truncated Lévy distribution of increasing width. All expansion coefficients c_n of $H(p)$ in Eq. (30) receive the same factor $t_b - t_a$, which has the consequence that the kurtosis $\kappa = c_4/c_2^2$ decreases inversely proportional to $t_b - t_a$. The distribution becomes increasingly Gaussian with increasing time, as a manifestation of the central limiting theorem*central+limiting+theorem* of statistical mechanics. This is in contrast to the pure Lévy distribution which has no finite width and therefore maintains its power falloff at large distances.

From the Fourier representation (52) it is easy to verify that this probability satisfies a Fokker-Planck-type equation

$$\partial_t (x_b t_b | x_a t_a) = -H(-i\partial_x)(x_b t_b | x_a t_a). \tag{54}$$

The general solution $\psi(x, t)$ of this differential equation with the initial condition $\psi(x, 0)$ is given by the path integral generalizing (44)

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$$\psi(x,t) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]\psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right). \tag{55}$$

To verify that this satisfies indeed the Fokker-Planck-type equation (54) we consider $\psi(x,t)$ at a slightly later time $t+\epsilon$ and expand

$$\psi(x,t+\epsilon) \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]\psi\left(x - \int_{t_a}^{t} dt'\eta(t') - \int_{t}^{t+\epsilon} dt'\eta(t')\right).$$

$$= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]\left\{\psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\right.$$

$$- \psi'\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\int_{t}^{t+\epsilon} dt'\eta(t')$$

$$+ \frac{1}{2}\psi''\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\int_{t}^{t+\epsilon} dt_1 dt_2\,\eta(t_1)\eta(t_2) \tag{56}$$

$$- \frac{1}{3!}\psi'''\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\int_{t}^{t+\epsilon} dt_1 dt_2 dt_3\,\eta(t_1)\eta(t_2)\eta(t_3)$$

$$\left.+ \frac{1}{4!}\psi^{(4)}\left(x - \int_{t_a}^{t} dt'\eta(t')\right)\int_{t}^{t+\epsilon} dt_1 dt_2 dt_3 dt_4\,\eta(t_1)\eta(t_2)\eta(t_3)\eta(t_4) + \dots\right\}.$$

Inserting here correlation functions (47) and dropping the vanishing odd expansion terms we obtain

$$\psi(x,t+\epsilon) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]\psi\left(x - \int_{t_a}^{t} dt'\eta(t') - \int_{t}^{t+\epsilon} dt'\eta(t')\right).$$

$$= \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]$$

$$\times \left[1 + \epsilon c_2 \frac{1}{2}\partial_x^2 + (\epsilon c_4 + \epsilon^2 c_2^2)\frac{1}{4!}\partial_x^4 + \dots\right]\psi\left(x - \int_{t_a}^{t} dt'\eta(t')\right).$$

The differential operators in the brackets can be pulled out of the integral and we obtain

$$\psi(x,t+\epsilon) = \int \mathcal{D}\eta \exp\left[-\int_{t_a}^{t_b} dt\,\tilde{H}(\eta(t))\right]\psi\left(x - \int_{t_a}^{t} dt'\eta(t') - \int_{t}^{t+\epsilon} dt'\eta(t')\right).$$

$$= \left[1 + \epsilon c_2 \frac{1}{2}\partial_x^2 + (\epsilon c_4 + \epsilon^2 c_2^2)\frac{1}{4!}\partial_x^4 + \dots\right]\psi(x,t). \tag{57}$$

In the limit $\epsilon \to 0$, only the linear terms in ϵ contribute, and we find the differential equation

$$\partial_t\psi(x,t) = \left[c_2\frac{1}{2}\partial_x^2 + c_4\frac{1}{4!}\partial_x^4 + \dots\right]\psi(x,t). \tag{58}$$

To lowest order in ϵ, only the connected parts of the correlation functions of $\eta(t)$ contribute. Comparison with the expansion (30) of the Hamiltonian (20) shows that the differential operators in brackets is precisely the Hamiltonian operator $-H(-i\partial_x)$, and we find

$$\partial_t\psi(x,t) = -H(-i\partial_x)\,\psi(x,t). \tag{59}$$

As an important side result of this calculation we note that the time derivative of an arbitrary function of the fluctuating variable $x(t)$ satisfying the stochastic differential equation (43) can be treated by the following generalization of Itô's rule

$$\dot{f}(x) = f'(x)\dot{x} + \left[\frac{1}{2}c_2\partial_x^2 + \frac{1}{4!}\partial_x^4 + \dots\right]f(x) = -H(-i\partial_x)f(x). \tag{60}$$

For a function $f(x) = e^{Px}$, this becomes simply

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$$\frac{d}{dt}e^{Px} = e^{Px}\dot{x} - H(-iP)e^x, \tag{61}$$

a result closely related to (42).

The above formalism is trivially generalized to processes with an average linear growth (7):

$$\dot{x}(t) = r_x + \eta(t). \tag{62}$$

Because of Eq. (60), however, the rate r_S with which the stock price $S(t) = e^{x(t)}$ grows is now related to r_x

$$r_x = r_S + H(-i), \tag{63}$$

instead of the simple Itô relation $r_x = r_S - \sigma^2/2$ of Eq. (8). The forward price of a stock must therefore be calculated with the generalization of formula (9):

$$\langle S(t) \rangle = S(0)\, e^{r_S t} = S(0)\, e^{[r_x - H(-i)]t}. \tag{64}$$

IV. CONCLUSION

The new stochastic calculus developed in this paper will be useful for estimating financial risks of a variety of investments. In particular, it will help developing a more realistic theory of fair option prices than available at present.

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