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Kosterlitz–Thouless-like deconfinement mechanism in the $(2+1)$-dimensional Abelian Higgs model

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Abstract

We point out that the permanent confinement in a compact $(2+1)$-dimensional $U(1)$ Abelian Higgs model is destroyed by matter fields in the fundamental representation. The deconfinement transition is Kosterlitz–Thouless-like. The dual theory is shown to describe a three-dimensional gas of point charges with *logarithmic* interactions which arises from an anomalous dimension of the gauge field caused by critical matter field fluctuations. The theory is equivalent to a sine-Gordon-like theory in $(2+1)$-dimensions with an *anomalous gradient energy* proportional to k^3. The Callan–Symanzik equation is used to demonstrate that this theory has a massless and a massive phase. The renormalization group equations for the fugacity $y(l)$ and stiffness parameter $K(l)$ of the theory show that the renormalization of $K(l)$ induces an anomalous scaling dimension η_y of $y(l)$. The stiffness parameter of the theory has a universal jump at the transition determined by the dimensionality and η_y. As a byproduct of our analysis, we relate the critical coupling of the sine-Gordon-like theory to an a priori arbitrary constant that enters into the computation of critical exponents in the Abelian Higgs model at the charged infrared-stable fixed point of the theory, enabling a determination of this parameter. This facilitates the computation of the critical exponent ν at the charged fixed point in excellent agreement with one-loop renormalization group calculations for the three-dimensional XY model, thus confirming expectations based on duality transformations.

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1. Introduction

Gauge theories in $d = 2 + 1$ dimensions are often considered as effective theories of strongly correlated systems in two spatial dimensions at zero temperature [1–3]. Prominent examples of systems to which such theories are hoped to be applicable are the high-T_c cuprates in the underdoped or undoped regime. In the undoped regime it is known that spinor QED_3 is an effective low energy theory for the quantum Heisenberg antiferromagnet (QHA) [1]. It is hoped that one effectively can account for doping by coupling the gauge theory to a scalar boson representing the holon part (charge part) of composite Hubbard-operators describing *projected* electrons, which however do not satisfy simple fermion commutation relations. Similar effective theories have a long history as useful toy-models in high-energy physics [4–6], and have recently been suggested to describe neural networks [7].

Of particular interest in the physics of strongly correlated systems is the compact version of the $(2 + 1)$-dimensional Abelian Higgs model with matter fields in the fundamental representation. This is the model we shall be concerned with in this paper and for which we shall find the results summarized in the abstract.

1.1. Preliminary remarks

Our starting point is the following Abelian euclidean field theory of a scalar matter field coupled to a massless gauge field

$$\mathcal{L}_b = \left|\left(\partial_\mu - i A^0_\mu\right)\phi_0\right|^2 + m_0^2|\phi_0|^2 + \frac{u_0}{2}|\phi_0|^4, \tag{1}$$

where the subscript zero denotes bare quantities. It corresponds to a theory with a Maxwell term

$$\mathcal{L}_M = \frac{1}{4e_0^2} {F^0_{\mu\nu}}^2, \tag{2}$$

where $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$, in which the gauge coupling e_0 goes to infinity. This limit implies the constraint $j_b^\mu = 0$, where $j_b^\mu = \phi_0^* \overset{\leftrightarrow}{\partial}{}^\mu \phi_0$ is the boson current.

When deriving *effective* theories for the t–J model we arrive naturally at a *compact* $U(1)$ lattice gauge field [2]. For QHA, the gauge symmetry is larger and given by the gauge group $SU(2)$ [3]. However, in this case a reduced $U(1)$ formulation is also possible [1]. Since this $U(1)$ is a subgroup of $SU(2)$, which is a compact group, the $U(1)$ gauge theory of QHA is necessarily a compact Abelian gauge theory.

It is well known that a compact $U(1)$ theory of the pure Maxwell type in three dimensions confines electric charges permanently [8]. In the literature [9] it is also argued that this permanent confinement should be present if an additional fermionic field ψ coupled to the gauge field by a Lagrangian

$$\mathcal{L}_f = \sum_{i=1}^{N} \bar{\psi}_i \left(\partial_\mu - i A^0_\mu\right)\psi_i . \tag{3}$$

This means that the particles represented by the fields ψ and ϕ_0 never have an independent dynamics. In the context of many-body theory, the Dirac fermion ψ could represent a *spinon*, while ϕ represents a *holon*. If electric test charges were permanently confined in the model, then the spinon and the holon would only appear as composite particles. In this case it would be impossible to fractionalize the electron, i.e. spin and charge would always remain attached to each other. Spin-charge separation is known to occur in $1+1$ dimensions [10]. There fermions can be transmuted into bosons via the so-called Jordan–Wigner transformation. In $2+1$ dimensions the situation is less clear, but for matter fields in the fundamental representation there is one circumstance where spin-charge separation is known rigorously to occur, namely the chiral spin liquid state [11]. However, the statistics of particles can be changed as in $1+1$ dimensions. In the chiral spin liquid, spinons have anyonic statistics described by a Chern–Simons term [12] in the effective gauge theory, which reflects the breaking of parity and time reversal symmetry.

The lack of consensus about spin-charge separation in $(2+1)$-dimensional compact $U(1)$ matter-coupled gauge theories with matter fields in the fundamental representation initiated investigations of other gauge theories for strongly correlated electron systems. One of the most promising candidates is a Z_2 gauge field coupled to matter fields [13]. Similar ideas leading to electron fractionalization had earlier been presented in the condensed matter literature [14,15]. In $2+1$ dimensions the Z_2 theory has a deconfinement transition [5]. Thus, Z_2 gauge theories are potentially good candidates for describing spin-charge separation without breaking parity and time reversal symmetries.

The confinement properties of $U(1)$ gauge theories for the cuprates and the relation to spin-charge separation were recently discussed from various points of view [9,16–19]. Nayak [9] states that in gauge theories of the t–J model fermions and bosons interact at infinite (bare) gauge coupling and, for this reason, it is necessarily a theory with permanent confinement of slave particles. In contrast, Ichinose and Matsui [18] have argued that the coupling to matter fields strongly influences the phase structure of the system. In Ref. [19], it is correctly pointed out that if spin-charge separation occurs, it is not necessarily tied to the notion of confinement–deconfinement of slave particles. The picture proposed in Ref. [9] in $2+1$ dimensions is reminiscent of $1+1$ dimensions where spinons and holons are solitons and cannot be identified with the slave particles, which are not part of the spectrum [10]. Nagaosa and Lee [17] discuss a compact $U(1)$ gauge theory coupled to bosonic matter field in the fundamental representation. They conclude that in $d=2+1$ this theory permanently confines electric charges, in contrast to the analysis by Einhorn and Savit on the same model [4].

In a recent letter [20], we have studied the confining properties of the Lagrangian (1), as well as the case of a fermionic field ψ coupled to a gauge field, but with an added Maxwell term. The Lagrangian (1) with a Maxwell term corresponds essentially to the model considered by Nagaosa and Lee [17], though these authors have considered a frozen-amplitude version of the model. In Ref. [20], it was emphasized that an anomalous scaling dimension of the gauge field, arising from matter-field fluctuations, changes the interaction between monopoles from $1/r$ to $\ln r$ in three dimensions. It was then argued that a monopole–antimonopole unbinding transition similar to the Kosterlitz–Thouless (KT) transition takes place, but now in three dimensions. From this, we concluded that test charges undergo a deconfinement transition.

It must be pointed out that the authors of Refs. [5,17], were looking for a transition similar to those encountered in $d = 3 + 1$, namely ordinary first- or second-order phase transitions [5]. In Ref. [17], a duality transformation was performed showing that the disorder parameter $\langle \phi_V \rangle$ is always different from zero, implying that $\langle \phi \rangle$ is always zero. *This result is essentially correct and is perfectly consistent with the scenario in* Ref. [20] and explained further in the present paper.

A main result in our letter [20] is that there exists a non-trivial infrared stable fixed point in the theory in $d = 2 + 1$ which drives the deconfinement transition. There the anomalous dimension of the gauge field is given by $\eta_A = 1$ in $d = 2+1$ [21,22]. *This result is exact as a consequence of gauge invariance. It implies that the non-trivial infrared fixed point arises at an infinite bare gauge coupling.* To see this, consider the boson–fermion Lagrangian $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_M$. Due to gauge invariance, the gauge coupling renormalizes to $e^2 = Z_A e_0^2$, where Z_A is the wave function renormalization constant of the gauge field. The renormalization group (RG) β function for the renormalized dimensionless gauge coupling $\alpha = e^2/\mu$ has the following exact form in $2 + 1$ dimensions

$$\beta_\alpha(\alpha, g) = \mu \frac{\partial \alpha}{\partial \mu} = \big[\gamma_A(\alpha, g) - 1\big]\alpha, \tag{4}$$

where g is the renormalized dimensionless $|\phi|^4$ coupling and $\gamma_A = \mu \partial \ln Z_A / \partial \mu$. Let us assume that there exist non-trivial infrared stable fixed points α_* and g_*, where the β functions β_α and β_g vanish. We have explained in Ref. [20] why such fixed points must exist. (For similar arguments, see Ref. [23].) Moreover, large-scale Monte Carlo simulations have demonstrated explicitly the existence of such a non-trivial fixed point [22,24] (see also Ref. [25]). Its existence has long been assured theoretically by duality arguments [26,27] (see also Section 2.2). We shall not repeat the arguments and details here. Instead, we focus on the physical consequences of the non-trivial fixed point.

We would like to stress an important point, pertinent to $d = 2 + 1$ dimensions, and quite different from the situation for $d = 3 + 1$. As $\alpha \to \alpha_*$, the bare coupling e_0^2 must tend to infinity. *By definition*, the above β function is given at fixed Λ, α_0, and g_0. Here, Λ is the ultraviolet cutoff while $\alpha_0 = e_0^2/\Lambda$ and $g_0 = u_0/\Lambda$ are the dimensionless *bare* couplings. The fixed point is reached for $\mu \to 0$. Alternatively, the fixed point is reached for $\Lambda \to \infty$ if μ is held fixed. However, since α_0 is fixed it follows that $e_0^2 \to \infty$ as $\Lambda \to \infty$. Thus, in $d = 2 + 1$, the fixed point theory is at *infinite bare gauge coupling*. One might object that this infinite gauge coupling cannot be relevant for the cuprates which have an infinite value of e_0^2 at *all* scales, not only in the scale invariant regime. This is true, but irrelevant as far as the deconfinement transition is concerned, which is determined by the non-trivial fixed point structure. The situation is analogous to the $O(N)$ non-linear σ model as opposed to the $O(N)$ ϕ^4 model. These models are quite different, but agree with each other at the critical point [28,29], thus belonging to the same universality class. In our case, the model with the Maxwell term at the fixed point has the same correlation functions as the model without it also at the fixed point.

To summarize the discussion in the above paragraph, the non-compact action with no Maxwell term has the same critical behavior as the compact one *at the critical point corresponding to a non-trivial fixed point, characterized by an infinite bare coupling*. Had we started from an infinitely weak bare coupling, the only fixed point we would have any hope of reaching for $d = 2 + 1$ would be the Gaussian fixed point.

In Ref. [20] we have pointed out that chiral symmetry breaking can destroy the deconfinement in the fermionic case. We want to point out that for the combined boson–fermion model, $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_b + \mathcal{L}_M$, chiral symmetry breaking does not spoil the deconfinement transition. Chiral symmetry breaking occurs at a lower value of number of fermion flavours N_f, when also bosons are present. Kim and Lee [30] claimed that the critical value of N_f is decreased by a factor two. Since we have typically $N_f^c \sim 3$ and the physical number of fermion components in the cuprates is $N_f = 2$, Kim and Lee argued that spin-charge separation would occur at finite doping [30].

1.2. Anomalous scaling and the potential between test charges

The high-energy physics literature is usually concerned with $d = 4$ and use low-dimensions only in toy models. In condensed matter physics, however, $(2+1)$-dimensional gauge theories are supposed to describe real physical phenomena such as the anomalous properties of high-T_c superconductors [31], or the physics of QHA [1,32]. For $d \in (2, 4]$ the gauge coupling β-function may be written as

$$\beta_\alpha(\alpha, g) = \left[\gamma_A(\alpha, g) + d - 4\right]\alpha. \tag{5}$$

Non-trivial fixed points induce an anomalous scaling behavior in the gauge field propagator. In the Landau gauge we have that

$$D_{\mu\nu}(p) = D(p)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right), \tag{6}$$

with the large distance behavior given by

$$D(p) \sim \frac{1}{|p|^{2-\eta_A}}. \tag{7}$$

The anomalous scaling dimension is given exactly by [21,22]

$$\eta_A \equiv \gamma_A(\alpha_*, g_*) = 4 - d. \tag{8}$$

Due to the above result, the propagator (7) in configuration space becomes

$$D(x) \sim \frac{1}{|x|^{d-2+\eta_A}} \sim \frac{1}{|x|^2}, \tag{9}$$

for all $d \in (2, 4]$. The potential between *effective electric charges* $q(R)$, separated by a large distance R in $(d-1)$-dimensional space is given by

$$V(R) \sim \frac{q^2(R)}{R^{d-3}}, \tag{10}$$

where

$$q^2(R) \sim \frac{1 - (\Lambda R)^{-\eta_A}}{\eta_A} \sim \frac{(\Lambda R)^{d-4} - 1}{d-4}, \tag{11}$$

and where Λ is a short distance cutoff. The anomalous scaling in Eq. (11) is a consequence of the coupling to matter fields. Due to it, the potential $V(R)$ behaves effectively like $1/R$ for $d = 3$. For $d = 4$, it goes like $\ln(\Lambda R)/R$, while for $d = 2$, it has a confining behavior proportional to R. The regime governed by the Gaussian fixed point has $q^2(R) = q_0^2 =$ const, and corresponds to the so-called Coulomb phase. In this phase, the four-dimensional theory has $V(R) = q_0^2/R$, whereas $V(R) = q_0^2 \ln R$ for $d = 3$. *We see that the non-trivial infrared behavior induces an effective electric potential between test charges similar to that which characterizes the Coulomb phase in $d = 4$.* If we extrapolate to $d = 2$, we obtain $V(R) = q_0^2 R$. Note that in $d = 2$, we obtain a confining potential irrespective of whether anomalous scaling is taken into account or not.

In compact Abelian gauge theories a confined phase is realized by the formation of *electric* flux tubes connecting electric charges. These flux tubes are the dual analogs of the *magnetic* flux tubes connecting magnetic monopoles [33,34]. There is a Dirac relation between the effective electric and magnetic charges

$$q(R)q_m(R) \sim 1. \tag{12}$$

Let us consider now the potential between the magnetic charges

$$V_m(R) \sim \frac{q_m^2(R)}{R^{d-3}} \sim \frac{1}{q^2(R)R^{d-3}}. \tag{13}$$

From Eq. (11) we see that for $d = 4$ the magnetic potential behaves like $1/[R\ln(\Lambda R)]$. However, for $d = 3$ we have

$$V_m(R) \sim \frac{1}{R}, \tag{14}$$

which is self-dual with respect to the potential between electric test charges.

The Higgs phase for the *electric charges* corresponds to $V(R) \sim$ const because of the gauge field mass gap. The Higgs phase for *magnetic test charges*, on the other hand, is given by $V(R) \sim R$. In the electric–magnetic duality picture [33,34] this Higgs phase for magnetic charges is exchanged by the confined phase for electric charges. This scenario should be valid for matter fields in the *adjoint* representation. In the absence of matter fields, a compact $(2 + 1)$-dimensional gauge theory is definitely confined permanently [8]. The above result shows that if matter fields are present, a deconfined phase is also possible. However, if the matter fields are in the fundamental representation, the situation is controversial [4,9,17–20]. Our recent results in Ref. [20] seem to be confirmed by the Monte Carlo work in Ref. [6]. The main purpose of this paper is to give more details on the scenario proposed in Ref. [20] and to describe a theory for a deconfinement transition in Abelian gauge theories coupled to matter fields in the fundamental representation.

1.3. Outline of the paper

In Section 2, we consider the lattice duality transformations to the $(2 + 1)$-dimensional Abelian Higgs lattice (AHL) model, first the non-compact case and later the compact case. We then discuss the possible ordinary first- or second-order phase transitions these models can have, with matter fields in the fundamental representation for the compact case.

In Section 3, we construct the continuum effective Lagrangian and its dual counterpart for the compact $(2+1)$-dimensional AHL model when matter-fields have been integrated out. Because these are central results of the paper, it behooves us to announce them here.

The dual field theory is given by Eq. (51). It represents a description of a three dimensional gas of point charges interacting with a *logarithmic* pair-potential, given by Eq. (49). We emphasize that the 3d ln-plasma action of Eq. (49) emerges from an underlying matter-coupled gauge theory, Eq. (38), by integrating out the fluctuating matter fields and considering the influence of *critical* matter fluctuations on the gauge-field propagator. The result of this procedure is the effective theory Eq. (46). Such matter-field fluctuations endow the gauge-field propagator with an anomalous scaling dimension $\eta_A = 4-d$ [21,22] which in three-dimensions alters the interaction between the monopole configurations of the gauge-field from a Coulomb-interaction $1/R$ to a $\ln R$ interaction.

Recall that in contrast to this, in the classic treatment by Polyakov [8] of compact three-dimensional QED with no matter fields, the standard three-dimensional sine-Gordon field theory with a quadratic gradient term, describing the three-dimensional Coulomb gas, is obtained. This action is given by, in the notation of Eq. (51)

$$S_{\mathrm{SG}} = \frac{1}{2t}\int d^3x\,\bigl[\varphi\bigl(-\partial^2\bigr)\varphi - 2z_0\cos\varphi\bigr]. \tag{15}$$

Polyakov has demonstrated [8] that Eq. (15) has no phase transition, i.e., it is always massive. Our Eq. (51) differs drastically from Eq. (15), due the presence of an anomalous gradient term.

In Section 4.1, we show using the Callan–Symanzik equations, that the effective dual Lagrangian Eq. (51) has a massless and a massive phase separated at a critical coupling t_c. Hence a phase transition must exist. This does not by itself suffice to show precisely *what sort of phase transition* the system undergoes, nor does it allow us to construct the correct flow diagram of the coupling constants of the problem. It does, however, suffice to show that two different phases exist. Since the propagator of the problem is logarithmic in $d = 2+1$, a Hohenberg–Mermin–Wagner theorem [36] holds. Under such circumstances, it is very natural to conjecture that any phase transition in the system, if it exists, must be of a *topological character*. In Section 4.2, we construct the renormalization group flow equations for the problem and show that the phase transition is of a KT-like type.

In Section 5, we consider the connection between the renormalization group functions obtained directly from the Abelian Higgs model, and the KT phase transition we find in Section 4. The main point here is that we can use the value of the critical coupling of the dual effective Lagrangian for the topological defects of the gauge field to fix an a priori arbitrary constant which enters into evaluating critical exponents for the *non-compact* Abelian Higgs model.

In Section 6, we conclude with a summary and outlook. Appendix A discusses another type of sine-Gordon theory also exhibiting a KT-like transition in three dimensions. In Appendix B, we derive the flow equations for the stiffness parameter and the fugacity of the system defined by Eq. (49), and of which Eq. (51) is a field theory formulation. In Appendix C, we compute the screened effective potential between charges in the insulating phase of the 3d ln-plasma. In Appendix D, for completeness, we derive the exact equation of state for a d-dimensional ln-plasma *with no short-distance cutoff* and relate the singularities in this plasma to the Callan–Symanzik approach of Section 4. In Appendix E, we consider, also for completeness, the duality transformation of the AHL model with a Chern–Simons term added. This case is of interest in the fractional quantum Hall effect [37] and chiral spin liquids [11].

2. Duality in the Abelian Higgs lattice model

In this section we review the duality approach to the AHL model. Although this is a well studied topic [4,26,27,38,39], it is worth reviewing it here in order to emphasize the differences and similarities between the non-compact and compact cases. In particular, we shall discuss the extent to which these cases exhibit ordinary first- or second-order phase transitions. The interesting case including a Chern–Simons term will be discussed in Appendix D.

The essential point is that starting from a non-compact or compact AHL model, the dual action has the general form

$$S_{\text{dual}} = \frac{1}{2}\sum_{i,j} h_{i\mu} M_{\mu\nu}(\mathbf{r}_i - \mathbf{r}_j) h_{i\nu} - i2\pi \sum_i \mathbf{l}_i \cdot \mathbf{h}_i, \tag{16}$$

where $h_{i\mu} \in (-\infty, \infty)$ and $\mathbf{l}_i$ are integer dual link variables. In the non-compact case $\mathbf{l}_i$ satisfy the constraint

$$\nabla \cdot \mathbf{l}_i = 0, \tag{17}$$

whereas in the compact case, the right-hand side is non-zero

$$\nabla \cdot \mathbf{l}_i = Q_i, \tag{18}$$

due to monopole charges $Q_i \in \mathbb{Z}$. The symbol ∇ denotes the gradient vector on a simple cubic lattice of unit spacing with components $\nabla_\mu f_i \equiv f_{i+\hat{\mu}} - f_i$.

2.1. *The non-compact case and the "inverted" XY transition*

In the non-compact case, the partition function of the AHL model is given by

$$Z = \sum_{\{n_{i\mu}\}} \int_{-\pi}^{\pi} \left[\prod_i \frac{d\theta_i}{2\pi}\right] \int_{-\infty}^{\infty} \left[\prod_{i,\mu} dA_{i\mu}\right] \exp(-S), \tag{19}$$

where the action S is given by the Villain approximation

$$S = \frac{\beta}{2}\sum_{i,\mu}(\nabla_\mu\theta_i - A_{i\mu} - 2\pi n_{i\mu})^2 + \frac{1}{2e^2}\sum_i(\nabla \times \mathbf{A}_i)^2. \tag{20}$$

Using the identity

$$\sum_{m=-\infty}^{\infty} e^{(-t/2)m^2 + ixm} = \sqrt{\frac{2\pi}{t}} \sum_{n=-\infty}^{\infty} e^{(-1/2t)(x-2\pi n)^2}, \tag{21}$$

following directly from Poisson's formula

$$\sum_{n=-\infty}^{\infty} F(n) = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dx\, F(x) e^{2\pi i m x}, \tag{22}$$

we obtain

$$Z = \int_{-\infty}^{\infty} \left[\prod_{i,\mu} dA_{i\mu}\right] \sum_{\{\mathbf{m}_i\}} \delta_{\nabla\cdot\mathbf{m}_i,0} \exp\left\{\sum_i \left[-\frac{1}{2\beta}\mathbf{m}_i^2 i + \mathbf{A}_i\cdot\mathbf{m}_i - \frac{1}{2e^2}(\nabla\times\mathbf{A}_i)^2\right]\right\}. \tag{23}$$

The Kronecker delta in Eq. (23) is generated by the θ_i integrations. Now we should integrate out the gauge field $\mathbf{A}_i$. The easiest way of performing this integration is by the introduction of an auxiliary field $\mathbf{h}_i$ such that the partition function can be rewritten as

$$Z = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\prod_{i,\mu} dA_{i\mu}\, dh_{i\mu}\, db_{i\mu}\right] \sum_{\{\mathbf{M}_i\}} \delta(\nabla\cdot\mathbf{b}_i)$$
$$\times \exp\left\{\sum_i \left[-\frac{1}{2\beta}\mathbf{b}_i^2 + i\mathbf{A}_i\cdot(\mathbf{b}_i - \nabla\times\mathbf{h}_i) - \frac{e^2}{2}\mathbf{h}_i^2 + 2\pi i\mathbf{M}_i\cdot\mathbf{b}_i\right]\right\}, \tag{24}$$

where a summation by parts has been done to replace $\mathbf{h}_i\cdot(\nabla\times\mathbf{A}_i)$ by $\mathbf{A}_i\cdot(\nabla\times\mathbf{h}_i)$, and we have used the Poisson formula (22) to replace the integer variables $\mathbf{m}_i$ by continuum variables $\mathbf{b}_i$, at the cost of an additional sum over integer variables $\mathbf{M}_i$. We may now integrate out $\mathbf{A}_i$ to obtain a delta function $\delta(\mathbf{b}_i - \nabla\times\mathbf{h}_i)$, after which also $\mathbf{b}_i$ can be integrated out $\mathbf{b}_i$, yielding

$$Z = \sum_{\{\mathbf{M}_i\}} \int_{-\infty}^{\infty} \left[\prod_{i,\mu} dh_{i\mu}\right] \exp\left\{-\sum_i \left[\frac{1}{2\beta}(\nabla\times\mathbf{h}_i)^2 + \frac{e^2}{2}\mathbf{h}_i^2 - 2\pi i\mathbf{M}_i\cdot(\nabla\times\mathbf{h}_i)\right]\right\}. \tag{25}$$

Summing the last term in the exponent by parts and going over to integer variables $\mathbf{l}_i = \nabla\times\mathbf{M}_i$, we obtain

$$Z = \sum_{\{\mathbf{l}_i\}} \int_{-\infty}^{\infty} \left[\prod_{i,\mu} dh_{i\mu}\right] \delta_{\nabla\cdot\mathbf{l}_i,0} \exp\left\{-\sum_i \left[\frac{1}{2\beta}(\nabla\times\mathbf{h}_i)^2 + \frac{e^2}{2}\mathbf{h}_i^2 - 2\pi i\mathbf{l}_i\cdot\mathbf{h}_i\right]\right\}. \tag{26}$$

Note that the Kronecker delta constraint above is a direct consequence of our change to integer-valued variables. If $\mathbf{h}_i$ is integrated out we obtain

$$Z = Z_0 \sum_{\mathbf{l}_i} \delta_{\nabla\cdot\mathbf{l}_i,0} \exp\left[-2\pi^2\beta \sum_{i,j,\mu} l_{i\mu} D(\mathbf{r}_i - \mathbf{r}_j) l_{j\mu}\right], \tag{27}$$

where the Green function G has the large-distance behavior

$$D(\mathbf{r}_i - \mathbf{r}_j) \sim \frac{e^{-\sqrt{\beta}\, e|\mathbf{r}_i - \mathbf{r}_j|}}{4\pi|\mathbf{r}_i - \mathbf{r}_j|}. \tag{28}$$

The factor Z_0 in Eq. (27) corresponds to the partition function of a free massive gauge boson theory.

Eq. (27) is the dual representation of the partition function for the non-compact AHL model. Due to the constraint $\nabla \cdot \mathbf{l}_i = 0$, the integer links $\mathbf{l}_i$ form closed loops.

By taking the limit $e \to 0$ in Eq. (23), we obtain

$$Z|_{e=0} = \sum_{\{\mathbf{m}_i\}} \delta_{\nabla\cdot\mathbf{m}_i,0} \exp\left(-\frac{1}{2\beta}\sum_i \mathbf{m}_i^2\right), \tag{29}$$

which is the loop gas representation of the XY model. If, on the other hand, we take the limit $\beta \to \infty$ in Eq. (27), we obtain the loop gas representation of the "frozen superconductor" [38]

$$Z|_{\beta=\infty} = \sum_{\{\mathbf{l}_i\}} \delta_{\nabla\cdot\mathbf{l}_i,0} \exp\left(-\frac{2\pi^2}{e^2}\sum_i \mathbf{l}_i^2\right), \tag{30}$$

which has precisely the same form as in Eq. (29). Therefore, the XY model is equivalent to the frozen superconductor, provided the Dirac-like relation $e^2 = 4\pi^2\beta$ holds. Eq. (27) is a reformulation of Eq. (19) in terms of the topological defects of the model, which are identified as integer-valued vortex strings forming closed loops.

If we consider the phase diagram in the $(e^2\text{–}T)$-plane (with $T = 1/\beta$), we can use Eqs. (29) and (30) to establish the critical points on the axes e^2 and T, corresponding to $T \to 0$ and $e^2 \to 0$ limits, respectively. From Eq. (29) we see that when $e^2 \to 0$ we have a XY critical point on the T-axis. Eq. (30) has exactly the same form as Eq. (29), but corresponds to the $T \to 0$ limit. The critical point in this limit is therefore $e_c^2 = 4\pi^2/T_c$, with T_c being the critical temperature of the XY transition as described by the Villain approximation. This is the so-called "inverted" XY transition (IXY) [26]. From the existence of these two critical points we can establish a phase diagram where there is a critical line connecting them [26]. The ordered superconducting phase corresponds to the region $0 < e^2 < e_c^2$.

2.2. The compact case and the absence of an ordinary phase transition

In the compact AHL model the gauge field $A_{i\mu} \in [-\pi, \pi]$. The corresponding Villain action is now given by

$$\widetilde{S} = \frac{\beta}{2}\sum_i (\nabla_\mu\theta_i - A_{i\mu} - 2\pi n_{i\mu})^2 + \frac{1}{2e^2}\sum_i (\epsilon_{\mu\nu\lambda}\nabla_\nu A_{i\lambda} - 2\pi N_{i\mu})^2, \tag{31}$$

and in the partition function we should sum over both integers $n_{i\mu}$ and $N_{i\mu}$. Using the identity (21) we obtain

$$Z = \sum_{\{\mathbf{n}_i\}}\sum_{\{\mathbf{m}_i\}} \int_{-\pi}^{\pi}\left[\prod_{i,\mu}\frac{dA_{i\mu}}{2\pi}\right]\int_{-\pi}^{\pi}\left[\prod_i \frac{d\theta_i}{2\pi}\right]\exp(S'), \tag{32}$$

where

$$S' = \sum_i \left[\frac{1}{2\beta}\mathbf{n}_i^2 + i\mathbf{n}_i \cdot (\nabla\theta_i - \mathbf{A}_i) + \frac{e^2}{2}\mathbf{m}_i^2 + i\mathbf{m}_i \cdot (\boldsymbol{\nabla} \times \mathbf{A}_i)\right]. \tag{33}$$

Now we integrate out $A_{i\mu}$ and θ_i to obtain

$$\begin{aligned} Z &= \sum_{\{\mathbf{n}_i\},\{\mathbf{m}_i\}} \delta_{\boldsymbol{\nabla}\cdot\mathbf{n}_i,0}\delta_{\boldsymbol{\nabla}\times\mathbf{m}_i,\mathbf{n}_i} \exp\left[-\sum_i\left(\frac{1}{2\beta}\mathbf{n}_i^2 + \frac{e^2}{2}\mathbf{m}_i^2\right)\right] \\ &= \sum_{\{\mathbf{m}_i\}} \exp\left[-\sum_i\left(\frac{1}{2\beta}(\boldsymbol{\nabla}\times\mathbf{m}_i)^2 + \frac{e^2}{2}\mathbf{m}_i^2\right)\right] \\ &= \sum_{\{\mathbf{l}_i\}} \int_{-\infty}^{\infty}\left[\prod_{i,\mu} dh_{i\mu}\right] \exp\left\{-\sum_i\left[\frac{1}{2\beta}(\boldsymbol{\nabla}\times\mathbf{h}_i)^2 + \frac{e^2}{2}\mathbf{h}_i^2 - 2\pi i\mathbf{l}_i\cdot\mathbf{h}_i\right]\right\}, \end{aligned} \tag{34}$$

where from the second to the third line we used the Poisson formula. Note the difference between Eq. (34) and its non-compact counterpart Eq. (26). In the latter there is a Kronecker delta constraint $\boldsymbol{\nabla}\cdot\mathbf{l}_i = 0$ while in the former there is no such a constraint. As we shall see, this difference has important consequences. We proceed by integrating out $h_{i\mu}$, thus obtaining the partition function

$$Z = Z_0 \sum_{\{\mathbf{l}_i\}} \exp\left[-2\pi^2\beta\sum_{i,j} l_{i\mu}D_{\mu\nu}(\mathbf{r}_i - \mathbf{r}_j)l_{j\nu}\right], \tag{35}$$

where

$$D_{\mu\nu}(\mathbf{r}_i - \mathbf{r}_j) = \left(\delta_{\mu\nu} - \frac{\nabla_\mu\nabla_\nu}{\beta e^2}\right)D(\mathbf{r}_i - \mathbf{r}_j), \tag{36}$$

$$\left(-\nabla^2 + \beta e^2\right)D(\mathbf{r}_i - \mathbf{r}_j) = \delta_{ij}. \tag{37}$$

Due to the constraint $\boldsymbol{\nabla}\cdot\mathbf{l}_i = 0$, the term containing $\nabla_\mu\nabla_\nu$ in Eq. (36) does not contribute in the non-compact case, and Eq. (27) results. In the compact case, on the other hand, $\boldsymbol{\nabla}\cdot\mathbf{l}_i$ is completely unconstrained and can take any integer value. Thus, in order to bring out the differences and similarities between Eqs. (35) and (27), and also to identify the character of the topological defects of Eq. (31) appearing in Eq. (35), we can introduce an auxiliary integer-valued scalar field Q_i such that $\boldsymbol{\nabla}\cdot\mathbf{l}_i = Q_i$ and rewrite the partition function (35) as

$$Z = Z_0 \sum_{\{\mathbf{l}_i\}}\sum_{\{Q_i\}} \delta_{\boldsymbol{\nabla}\cdot\mathbf{l}_i,Q_i} \exp\left[-2\pi^2\beta\sum_{i,j} D(\mathbf{r}_i - \mathbf{r}_j)\left(l_{i\mu}l_{j\mu} + \frac{1}{e^2\beta}Q_iQ_j\right)\right]. \tag{38}$$

Whereas the non-compact theory has only closed vortex lines as topological defects, the compact case contains also open lines with integer-valued monopoles of charge Q_i at the ends.

In the limit $\beta \to 0$, Eq. (38), the vortex loops are frozen out and (38) is the dual representation of three-dimensional lattice compact QED [8] describing a Coulomb gas of monopoles in three dimensions. This is equivalent to a sine-Gordon model which is always massive in three dimensions, and leads to the well-known result that compact QED in three dimensions has permanent confinement of electric charges, since the monopole gas will always be in the plasma phase. As shown by Polyakov [8], we obtain as a consequence that the Wilson loop satisfies the area law.

As in the non-compact case, the limit $e^2 \to 0$ corresponds to the Villain form of the XY model. Thus, if we consider again a phase diagram in the (e^2, T)-plane we have that a critical point at T_c exists on the T-axis. However, as we shall now show, *there is no IXY transition in the compact case*. To see this, let us take the "frozen" limit $\beta \to \infty$ in Eq. (38). The result is

$$Z = Z_0 \sum_{\{\mathbf{l}_i\}} \sum_{\{Q_i\}} \delta_{\nabla \cdot \mathbf{l}_i, Q_i} \exp\left(-\frac{2\pi^2}{e^2} \sum_i \mathbf{l}_i^2 \right). \tag{39}$$

The sum over Q_i is trivially done, $\sum_{\{Q_i\}} \delta_{\nabla \cdot \mathbf{l}_i, Q_i} = 1$ after which there is no constraint. We are left with a trivial sum over $\mathbf{l}_i$ giving Jacobi ϑ-functions $\vartheta_3(0, e^{-2\pi^2/e^2})$. Since this function is analytic, there is no phase transition on the e^2-axis, in contrast to the non-compact case. Thus, at first sight it seems that there is no phase transition in the compact AHL model with matter fields in the fundamental representation, except for the XY transition on the T-axis. That is, there appears to be no ordinary second- or first-order phase transition in the interior of the phase diagram of this model. However, in the next sections we shall derive an effective Lagrangian for the compact Abelian Higgs model in $2 + 1$ dimensions, which will be shown to nevertheless exhibit a topological phase transition of the KT type.

3. Effective Lagrangian

This section is one of the central parts of the paper in which we shall derive an effective field theory for the compact Abelian Higgs model in $d = 2+1$ dimensions. More precisely, we derive a continuum action, Eq. (51) below, for the dual model of the system, obtained after matter fields have been integrated out leaving an effective theory for the monopoles of the problem. It will turn out that the effective dual Lagrangian for the $(2+1)$-dimensional compact Abelian Higgs model, is described by a theory which has many similarities to the sine-Gordon theory of Polyakov's pure compact electrodynamics in $d = 2 + 1$ [8]. The crucial difference lies in the fact that the gradient term in the dual theory receives an anomalous dimension after the matter-fields have been integrated out. It is the presence of this anomalous gradient term induced by matter-field fluctuations which eventually will lead to the possibility of a deconfinement transition in $d = 2+1$, in contrast to the classical Polyakov result of permanent confinement pertaining to the pure gauge theory.

3.1. Three-dimensional compact QED

Let us consider the Euclidean Maxwell action in three dimensions:

$$S = \int d^3x \, \frac{1}{4e^2} F_{\mu\nu}^2, \tag{40}$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. In order to account for monopoles, we have to subtract from $F_{\mu\nu}$ the gauge field of monopoles [35]

$$F^M_{\mu\nu}(x) = 2\pi\epsilon_{\mu\nu\lambda}\delta_\lambda(x; L), \tag{41}$$

where $\delta_\lambda(x; L)$ is a delta function on lines L. The dual field strength of $\widetilde{F}^M_\lambda = \epsilon_{\mu\nu\lambda}F^M_{\mu\nu}/2$ has divergences at the end points of the lines L, say [34,35]

$$\partial_\mu \widetilde{F}^M_\mu = 2\pi n(x) = 2\pi \sum_i Q_i \delta^3(x - x_i), \tag{42}$$

where Q_i may be arbitrary are integers counting the number of lines ending at x_i. The shape of the lines is physically irrelevant. They are the Dirac strings of the monopoles at x_i. Under shape deformations, $F^M_{\mu\nu}$ undergoes the monopole gauge transformations $F^M_{\mu\nu} \to F^M_{\mu\nu} + \partial_\mu \Lambda^M_\nu - \partial_\nu \Lambda^M_\mu$ which leave $\widetilde{F}^M_\mu$ invariant.

An ordinary gauge transformation can be used to bring $F^M_{\mu\nu}(x)$ to the form

$$F^M_{\mu\nu} = -2\pi\epsilon_{\mu\nu\lambda}\partial_\lambda \int d^3y \, \frac{1}{4\pi|x-y|} n(y), \tag{43}$$

whose dual field strength is

$$\widetilde{F}^M_\mu = -2\pi\partial_\mu \int d^3y \, \frac{1}{4\pi|x-y|} n(y). \tag{44}$$

By substituting $F_{\mu\nu}$ by $F_{\mu\nu} - F^M_{\mu\nu}$ in Eq. (40), we obtain the action

$$S = \int d^3x \, \frac{1}{4e^2} F^2_{\mu\nu} + \frac{2\pi^2}{e^2} \int d^3x \int d^3y \, n(x) \frac{1}{4\pi|x-y|} n(y). \tag{45}$$

The action (45) corresponds to the continuum counterpart of the $\beta \to 0$ limit of the lattice action in Eq. (38) describing a Coulomb gas of monopoles. This is known to be equivalent to a sine-Gordon action as the one in Eq. (15). In three dimensions this theory is always massive and it was shown by Polyakov [8] that this implies an area law for the Wilson loop. Thus, electric test charges in three-dimensional compact QED are permanently confined.

3.2. *Anomalous three-dimensional compact QED*

When bosonic matter fields are present, the topological defects of the theory are vortex loops and vortex lines having monopoles with opposite charges at the ends. The vortex lines connecting the monopoles have a line tension σ which vanishes as the scalar bosons become massless. Thus, when the vortex lines become tensionless, we are left with a gas of monopoles. However, the anomalous scaling of the gauge field due to matter fields alters the interaction between pair of monopoles with respect to the ordinary Coulomb interaction case. This will lead us to the *anomalous* Coulomb gas to be described below.

From the exact behavior of the critical gauge field propagator we have discussed in Section 1.2, we can write an effective quadratic *non-local* Lagrangian for the gauge field:

$$\mathcal{L}_A = \frac{K}{4} F_{\mu\nu} \frac{1}{(-\partial^2)^{\eta_A/2}} F_{\mu\nu}$$
$$= \frac{K}{2} \widetilde{F}_\mu \frac{1}{(-\partial^2)^{\eta_A/2}} \widetilde{F}_\mu, \tag{46}$$

where the constant $K = K(\alpha_*, g_*)$ and in the second line of Eq. (46) we have rewritten $\mathcal{L}_A$ in terms of the dual field strength. Specializing to three dimensions, we have $\widetilde{F}_\mu = \epsilon_{\mu\nu\lambda} F_{\nu\lambda}/2$ and $\eta_A = 1$. After introducing an auxiliary vector field b_μ, we obtain the equivalent Lagrangian:

$$\mathcal{L}'_A = \frac{1}{2K} b_\mu \sqrt{-\partial^2}\, b_\mu + i b_\mu \widetilde{F}_\mu. \tag{47}$$

In order to take into account the monopoles, we use the expression for $\widetilde{F}_\mu$ as given in Eq. (44). By introducing a new field through $b_\mu = \partial_\mu \varphi$ and using integration by parts, we obtain

$$\mathcal{L}''_A = \frac{1}{2K} (\partial_\mu \varphi) \sqrt{-\partial^2}\, (\partial_\mu \varphi) + i 2\pi n(x) \varphi(x). \tag{48}$$

Integrating out φ and using Eq. (42), we obtain the monopole action

$$S_{\text{mon}} = 2\pi^2 K \sum_{i,j} Q_i Q_j G(x_i - x_j), \tag{49}$$

where

$$G(x) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik\cdot x}}{|k|^3}. \tag{50}$$

Thus, instead of having a standard three-dimensional Coulomb gas with interaction potentials $1/|x_i - x_j|$, we have a three-dimensional gas of point particles of charge $Q_i = \pm|Q|$ (with overall charge-neutrality, see Section 4.1) *with logarithmic interactions*, much akin to the situation one has in two dimensions. We emphasize, once more, that this is a result of integrating out matter-field fluctuations and considering the effect of *critical* such fluctuations on the gauge-field propagator, which is seen to acquire an anomalous scaling dimension from these fluctuations, cf. Eq. (46). It therefore seems *plausible*, at the very least, that one should consider the possibility of having a KT-transition of unbinding of monopole–antimonopole pairs, but now in three dimensions. If this turns out to be the case, then the confinement–deconfinement transition in the $(2+1)$-dimensional compact Abelian Higgs model with matter fields in the fundamental representation, would be of a *topological* nature with no local order parameter, consistent with previous work [5,17].

We are now ready to state one of the main results of this paper. The system defined by Eqs. (46) and (49) can be brought into the form of a sine-Gordon theory, as in the two-dimensional case, *but now with an anomalous propagator*, whose action is

$$S_{\text{ASG}} = \frac{1}{2t} \int d^3x \left[\varphi\left(-\partial^2\right)^{3/2} \varphi - 2z_0 \cos\varphi\right], \tag{51}$$

where $t = 4\pi^2 K$ and $z_0 = 4\pi^2 K \zeta$, with ζ being the fugacity of the Coulomb gas of monopoles. In Eq. (51), S_{ASG} refers to the action of what we name the anomalous sine-Gordon (ASG) theory, since the cubic power of the propagator arises from the anomalous scaling dimension of the gauge field. The manner in which the coupling constant t enters in Eq. (51) shows that it regulates the stiffness of the phase field φ. Since $t \propto K$, we shall in following sections refer to K as a stiffness parameter.

4. Renormalization group analysis of the anomalous sine-Gordon model

This section is another central part of the paper. Here, we shall consider an exact scaling argument applied to Eq. (51). The scaling argument will suffice to demonstrate that this model has a phase transition. We emphasize this as an important point, since recent numerical studies [6] have provided strong support for the picture proposed in Ref. [20] that matter-field coupled to compact $U(1)$ gauge fields in $d = 2 + 1$ lead to a recombination of magnetic monopoles into dipoles. For the dual electric charges, this leads to a destruction of permanent confinement, and in Ref. [20] it was argued that this happened, not through any ordinary first- or second-order phase transition, but rather through a KT-like transition. The authors of Ref. [6] were looking for more conventional phase transitions, concluding that none were found, consistent with the results of Ref. [20]. Having established the existence of a phase transition, we then go on to argue that it indeed is of a KT-like type. The details are as follows.

4.1. Callan–Symanzik renormalization group analysis

Let us consider the renormalization of the anomalous sine-Gordon action defined by Eq. (51). The infrared divergence is easily studied by considering the cubic propagator $G(p) = 1/|p|^3$ in real space. To this end, we introduce an infrared cutoff μ as follows

$$G_\mu(x) = \int\limits_{|p|>\mu} \frac{d^3p}{(2\pi)^3} \frac{e^{ip\cdot x}}{|p|^3} = \frac{1}{2\pi^2}\left[\frac{\sin(\mu|x|)}{\mu|x|} - \mathrm{ci}\big(\mu|x|\big)\right], \tag{52}$$

where $\mathrm{ci}(\lambda)$ is the cosine integral

$$\mathrm{ci}(\lambda) \equiv -\int_\lambda^\infty \frac{\cos v}{v}\, dv. \tag{53}$$

As $\mu \to 0$ we have

$$G_\mu(x) = \frac{1}{2\pi^2}\big[1 - \gamma - \ln\big(\mu|x|\big)\big] + \mathcal{O}(\mu), \tag{54}$$

where γ is the Euler–Mascheroni constant. For $x = 0$, on the other hand, $G_\mu(x)$ is ultraviolet divergent and becomes

$$G_\mu(0) = \frac{1}{2\pi^2}\ln\left(\frac{\Lambda}{\mu}\right) + \mathrm{const} + \mathcal{O}\left(\frac{1}{\Lambda}\right), \tag{55}$$

where Λ is an ultraviolet cutoff.

Let us consider now the correlation function

$$\left\langle \prod_{j=1}^{n} e^{iq_j\varphi(x_j)} \right\rangle = \frac{1}{Z_0}\int \mathcal{D}\varphi \exp\left\{-\frac{1}{2t}\int d^3x\left[\int d^3y\,\varphi(x)G_\mu^{-1}(x-y)\varphi(y) - J(x)\varphi(x)\right]\right\}, \tag{56}$$

where $J(x) = i\sum_j q_j\delta(x-x_j)$ and Z_0 is the above functional integral for $J=0$. Integrating out φ, we obtain

$$\left\langle \prod_{j=1}^{n} e^{iq_j\varphi(x_j)} \right\rangle = \exp\left[-\frac{1}{2}\sum_{i,j} G_\mu(x_i-x_j)q_iq_j\right]. \tag{57}$$

Using Eqs. (54) and (55), we obtain

$$\sum_{i,j} G_\mu(x_i-x_j)q_iq_j = -\frac{1}{2\pi^2}\left[\left(\sum_i q_i\right)^2(\ln\mu+\gamma-1) - \sum_i q_i^2\ln\Lambda + \sum_{i\neq j} q_iq_j\ln|x_i-x_j|\right] + \mathcal{O}(\mu). \tag{58}$$

Thus, as $\mu\to 0$ the only non-zero contributions to (57) satisfy the neutrality condition for the charge $\sum_i q_i = 0$. The expansion in Eq. (58) is essentially the same as in the $d=2$ case, except for the $1/2\pi^2$ factor instead of a $1/2\pi$, and minor differences in the constants.

The ultraviolet divergence of the phase field $u_i(x)\equiv e^{iq_i\varphi(x)}$ is removed by introducing a wave function renormalization ζ_i such that

$$u_i(x) = \zeta_i^{1/2}u_{i,R}(x), \tag{59}$$

with $u_{i,R}$ being the renormalized counterpart of u_i and

$$\zeta_i = \left(\frac{\Lambda}{\mu}\right)^{-q_i^2/(2\pi^2)}. \tag{60}$$

Therefore, if we specialize to the case where $q_i = \pm|q|$ for all i, the renormalized two-point correlation function is given by

$$\left\langle u_{i,R}(x)u_{i,R}^\dagger(0)\right\rangle \propto x^{-q^2/(2\pi^2)}. \tag{61}$$

It follows that the dimension of u_i is just $q^2/(4\pi^2)$.

Due to the above analysis it is now easy to see how z_0 renormalizes in the ASG model. Note that the model is super-renormalizable, just as the ordinary sine-Gordon model. Thus, the renormalization of z_0 is achieved by taking into account only tadpole contractions of $\cos\varphi$. We obtain

$$z_0 = Z_\varphi^{-1/2}z, \tag{62}$$

where

$$Z_\varphi = \left(\frac{\Lambda}{\mu}\right)^{-t/(2\pi^2)}. \tag{63}$$

Furthermore, we have the RG function

$$\eta_\varphi \equiv \mu \frac{\partial \ln Z_\varphi}{\partial \mu} = \frac{t}{2\pi^2}. \tag{64}$$

The renormalized n-point correlation function $\mathcal{G}^{(n)}$ satisfies the following Callan–Symanzik equation

$$\left(\mu\frac{\partial}{\partial\mu} + \frac{n}{2}\eta_\varphi + \frac{1}{2}\eta_\varphi z\frac{\partial}{\partial z}\right)\mathcal{G}^{(n)}(p_i, t, z) = 0. \tag{65}$$

Dimensional analysis, on the other hand, gives

$$\left[\mu\frac{\partial}{\partial\mu} + 3z\frac{\partial}{\partial z} + p_i\frac{\partial}{\partial p_i} + 3(n-1)\right]\mathcal{G}^{(n)}(p_i, t, z) = 0, \tag{66}$$

where $3(1-n)$ represents the mass dimension of $\mathcal{G}^{(n)}$. Using Eq. (66) in (65), we obtain

$$\left[p_i\frac{\partial}{\partial p_i} + 3(n-1) - \frac{n}{2}\eta_\varphi + \left(3 - \frac{1}{2}\eta_\varphi\right)z\frac{\partial}{\partial z}\right]\mathcal{G}^{(n)}(p_i, t, z) = 0. \tag{67}$$

For $p = 0$ we have

$$(6-\eta_\varphi)z\frac{\partial\mathcal{G}^{(n)}(0,t,z)}{\partial z} = \left[6(1-n) + n\eta_\varphi\right]\mathcal{G}^{(n)}(0,t,z), \tag{68}$$

which gives the following scaling relation for small z

$$\mathcal{G}^{(n)}(0,t,z) \sim z^{[6(1-n)+n\eta_\varphi]/(6-\eta_\varphi)}. \tag{69}$$

Also, it is clear from Eq. (67) that the scaling behavior of the mass scale is

$$m_\varphi \sim z^{2/(6-\eta_\varphi)}. \tag{70}$$

The momentum space behavior of $\mathcal{G}^{(2)}$ is $\sim 1/p^{3-\eta_\varphi}$ and therefore $\mathcal{G}^{(2)}$ becomes singular in the ultraviolet if $3 - \eta_\varphi < 0$. This happens for $t = 6\pi^2$. For $t = 6\pi^2$ the mass scale behaves like $z^{2/3}$. This is an important difference between the usual sine-Gordon model in two dimensions and the CPSG model in three dimensions. The mass scale in the usual two-dimensional sine-Gordon theory behaves linearly in z when the singular short-distance behavior is reached. There, this behavior is important for the fermionization of the model, which establishes the equivalence between the sine-Gordon model and the Thirring model in two dimensions [40].

From Eqs. (69) and (70) we see that $\mathcal{G}^{(n)}(0,t,z)$ and m_φ vanish for $t = t_c = 12\pi^2$. The interpretation of this result closely parallels the one in the usual sine-Gordon model. For instance, it tells us that at $t = t_c$ the operator $\cos\varphi$ is marginal, and means that further renormalizations are necessary at $t = t_c$. The situation exactly parallels the two-dimensional case where a thorough analysis was carried out by Amit et al. [41]. For $t > t_c$

the anomalous sine-Gordon model Eq. (51) is no longer renormalizable. These results follow from the observation that the dimension of the operator $\cos\varphi$ is just $\eta_\varphi/2$. Thus, $\int d^3x\cos\varphi$ has dimension $\eta_\varphi/2-3$, which means that z has an effective dimension of $3-\eta_\varphi/2$. Therefore, the interaction is relevant for $\eta_\varphi<6$ or $t<t_c$, thus generating a mass. It is marginal if $\eta_\varphi=6$ and irrelevant for $\eta_\varphi>6$, or $t>t_c$, meaning that the theory is massless. Hence, there is a phase where the field has a mass and another one where it is massless, implying the existence of a genuine phase transition in the model Eq. (51). This follows from the fact that a mass changing from a finite value to zero on a finite interval of coupling constants must do so in a non-analytic fashion. This conclusion is one of the main results of this paper. Note, however, that since the above discussion is basically a spin wave analysis and suffices to show that a phase transition exists, it does not elucidate the *character* of the phase transition. In order to understand the phase transition we have to account for the topological defects in the theory [42], and this is the purpose of the next subsection.

4.2. *Kosterlitz–Thouless-like recursion relations for the anomalous sine-Gordon model*

The above discussion strongly suggests the existence of a phase transition in the model defined by Eq. (51). However, as we have already mentioned, the cosine interaction becomes marginal at $t=t_c$. This means that it is not true that $\beta(t)=0$ for all values of t. The analysis of the previous subsection is neglecting the monopole fluctuations which would lead to a renormalization of t. This situation is well known for the logarithmic interaction in two dimensions and leads to the KT-recursion relations [42,43]. Similar arguments can be used in our case.

Let us define the dimensionless coupling $y=z/\mu^3$. Using Eq. (62) we obtain the flow equation

$$\mu\frac{\partial y}{\partial\mu}=\left(\frac{t}{4\pi^2}-3\right)y. \tag{71}$$

The above equation can be derived in another way, which is useful for the purposes of this subsection. Let us consider again the monopole action in Eq. (49). We can write the partition function of the monopoles as

$$Z_{\rm mon}=\sum_{\{n(x)\}}\exp\left[-2\pi^2K\int_{|x_i|>a}d^3x\int_{|x_i'|>a}d^3x'\,n(x)G(x-x')n(x')\right], \tag{72}$$

where a is a short distance cutoff. Using Eqs. (54) and (55), we rewrite the above in the following form

$$Z_{\rm mon}=\sum_{\{n(x)\}}{}'\exp\left[-2\pi^2K\int_{|x_i|>a}d^3x\int_{|x_i'|>a}d^3x'\,n(x)\widetilde{G}(x-x')n(x')\right.$$
$$\left.+\ln y_0\int_{|x_i|>a}d^3x\,n^2(x)\right], \tag{73}$$

where

$$\widetilde{G}(x) = -\frac{1}{2\pi^2}\ln\frac{|x|}{a}. \tag{74}$$

The prime on the summation sign in Eq. (73) indicates that the charge neutrality constraint implied by the large distance limit is enforced. If we assume small y_0 such that configurations with zero or one pair of monopoles are dominant, we obtain

$$Z_{\text{mon}} \approx 1 + y_0^2 \int\limits_{|x_i|>a} d^3x \int\limits_{|x_i'|>a} d^3x' \frac{1}{|x - x'|^{2K}}. \tag{75}$$

If we change the short-distance cutoff in the integrals as $a \to ab$, we see that the form of Eq. (75) is unchanged provided x and x' are rescaled in such a way as to restore the previous integration region and y_0 is changed according

$$y = y_0 b^{3-K}. \tag{76}$$

If we define $l \equiv \ln b$, we obtain

$$\frac{dy}{dl} = (3 - K)y. \tag{77}$$

Recalling that $t = 4\pi^2 K$, we see that Eq. (77) is precisely Eq. (71), except for the sign, which is due to differences in the cutoff procedure. Eq. (77) is analogous to the corresponding flow equation for the fugacity in the ordinary KT transition [42,43]. In that case we find instead $dy/dl = (2 - \pi K)y$. The factor 2 in the usual KT case reflects the dimensionality. In our case we have a factor 3 instead (and also just K rather than πK).

It is also possible to derive recursion relations involving the fugacity of the problem in arbitrary dimensions to lowest orders in the fugacity for the d-dimensional Coulomb gas with a power law interaction

$$V(x) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{(4\pi)^{d/2}}\left[\left(\frac{|x|}{a}\right)^{2-d} - 1\right]. \tag{78}$$

This problem was considered by Kosterlitz [44], who also obtained the flow of the stiffness. The result is

$$\frac{dK^{-1}}{dl} = y^2 - (2 - d)K^{-1}, \tag{79}$$

$$\frac{dy}{dl} = \left[d - 2\pi^2 f(d)K\right]y, \tag{80}$$

where $f(d) = (d - 2)\Gamma[(d - 2)/2]/(4\pi)^{d/2}$. For $d = 2$ this reduces to the KT flow equations.

However, we see that for $d = 3$ the recursion relations (79) and (80) do not have a fixed point and therefore no phase transition happens in this case. In case of Eq. (49), on the other hand, we have an anomalous Coulomb gas whose potential is logarithmic in three dimensions. It is thus plausible to conjecture that we would have a flow equation for the stiffness similar to the $d = 2$ KT case. As we shall see, this is indeed the case.

For $d = 3$, Eq. (80) coincides with our Eq. (77). However, since the potentials are different we should in fact not expect the same recursion relation for the fugacity. This suggests that the "spin wave" picture of Section 4.1 is not giving the correct flow for the fugacity. Note that for $d = 2$ discussed in [41], *the spin wave analysis does in fact give the correct flow for the fugacity* (see Appendix B). For a logarithmic potential in $d = 3$ there are some subtleties.

Let us consider a problem with a potential like

$$V(x) = \frac{\Gamma\left(\frac{d-2-\eta_A}{2}\right)}{2^{\eta_A}(4\pi)^{d/2}\Gamma\left(\frac{2+\eta_A}{2}\right)}\left[\left(\frac{|x|}{a}\right)^{2-d+\eta_A} - 1\right]. \tag{81}$$

Here we have taken into account the effect of anomalous scaling due to matter-field fluctuations in our original problem. A logarithmic interaction corresponds to the case $d = 3$ and $\eta_A = 1$, which is the case which eventually will be relevant for us. Strictly speaking, the duality scenario in Section 3 is valid only at $d = 3$. However, as far as the scaling behavior is concerned, it is useful to continue to the whole dimension interval $(2, 4)$, while keeping the same ϵ-tensors. This dimensional continuation procedure is reminiscent of the one considered in some RG studies of Chern–Simons theories [45]. The recursion relations we obtain are given by (see Appendix B)

$$\frac{dK^{-1}}{dl} = y^2 - (2 - d + \eta_A)K^{-1},$$
$$\frac{dy}{dl} = \left[d - \eta_y - 2\pi^2 \tilde{f}(d)K\right]y, \tag{82}$$

where η_y is the *anomalous dimension* of the fugacity which is given by

$$\eta_y = \frac{\eta_A}{2} = \frac{4-d}{2}, \tag{83}$$

and

$$\tilde{f}(d) = \frac{(d-2-\eta_A)\Gamma\left(\frac{d-2-\eta_A}{2}\right)}{2^{\eta_A}(4\pi)^{d/2}\Gamma(1+\eta_A/2)}. \tag{84}$$

Hence, for the case of a *logarithmic interaction in three dimensions*, which corresponds to $\eta_A = 1$, the recursion relations for the fugacity and the stiffness have a similar structure as the standard Kosterlitz–Thouless recursion relations one obtains in the two-dimensional case [42,43]. The main difference is in the recursion relation for the fugacity, which has an anomalous dimension $\eta_y = 1/2$. Note that the second term in the equation for $K^{-1}(l)$, which prevents fixed points of Eqs. (82) from being obtained, is absent for a logarithmic potential in any dimension.

When $\eta_A = 0$, which corresponds to neglecting the effect of matter fields in the original gauge theory, we have $\eta_y = 0$. Our recursion relations then reduce to the ones given in Eqs. (79) and (80) obtained in [44] by a very different method than we employ in Appendix B. Moreover, we have also derived Eqs. (82) along a different route than that used in Appendix B, namely by the method employed in [43]. This constitutes an important consistency check on our calculations. For the case where $\eta_A = 0$, the absence of a phase transition reflects the permanent confinement of electric test charges in the usual three-dimensional compact QED [8].

We see that the flow equation for the fugacity obtained in Eq. (82) does not agree with the result of our "spin-wave" theory, which leads to Eq. (71) or, equivalently, Eq. (77). *The reason for this is that an anomalous scaling dimension η_y for the fugacity is induced by the renormalization of the stiffness*. Indeed, in Appendix B we show that a potential like (78) leads to an additional scaling transformation in the *effective* stiffness of the form $K(l) \to e^{(2-d+\eta_A)l} K(l)$. If $\eta_A \neq 0$, this is compensated in the *effective* fugacity by the scaling transformation, $y(l) \to e^{-\eta_y l} y(l)$. In the case of the Coulomb gas, where $\eta_A = 0$, the spin-wave analysis gives the right answer, Eq. (80), as can easily be seen by working out a Callan–Symanzik RG analysis in the sine-Gordon theory (15) for arbitrary dimensions. Thus, deviations from an ordinary type of Coulomb potential in d-dimensions lead to an anomalous dimension to the fugacity, Eq. (83), which cannot be obtained by spin-wave theory.

The important point to note here is that a fixed point of the recursion relations Eqs. (82) for $d = 3$ exists for the stiffness and fugacity in the limit of zero fugacity, so the problem scales to the weak coupling limit. Hence, the problem is selfconsistently found to be amenable to a KT-type of phenomenological RG analysis. It is not necessary to calculate to higher-order in y to determine the fixed point. This demonstrates that the phase transition established above is of the KT type. This has some resemblance with the results of a rather remarkable paper by Amit et al. [46], which also finds a KT transition in a three-dimensional Coulomb gas with logarithmic interaction between point charges (see Appendix A). In their case, the logarithmic interaction between the point charges in three dimensions did not have its origin in anomalous scaling dynamically generated by matter-field fluctuations, but originated in anisotropic higher-order derivative terms in an underlying field theory that were put in by hand. This anisotropy ultimately induces dimensional reduction.

In four dimensions, we have $\eta_A = 0$ and extrapolating the above results it is clear that no fixed points of the above recursion relations can be found. Indeed, the above analysis no longer applies and no KT topological phase-transition occurs. This is so because by dualizing a compact Maxwell Lagrangian in four dimensions, we obtain a non-compact Abelian Higgs model [38], which cannot be brought onto the form of a Coulomb gas. The transition in this case is known to be of more conventional second- or first-order type [5].

Finally, we note that in three dimensions there is a universal jump in the stiffness parameter at the transition, analogous to what is known in the 2d case [47]. In units of Eqs. (82), this jump is determined by dimensionality and the anomalous scaling of the fugacity,

$$K_R \equiv \lim_{l\to\infty} K(l) = \frac{d - \eta_y}{2\pi^2 \tilde{f}(d)}. \tag{85}$$

5. RG functions of Abelian Higgs model and KT transition

In this section we show how the RG functions and fixed points in the Abelian Higgs model are related to the KT-like transition described in the previous section. In particular, we shall use the critical coupling t_c to fix an a priori arbitrary constant that enters into the computation of the critical exponents for the Abelian Higgs model. This in our view improves on a scheme previously used [21], where a corresponding constant was fixed by appealing to numerical results for the value of the Ginzburg–Landau parameter κ which separates first- from second-order behavior.[1] In our approach, the parameter (denoted r below) is fixed from our theory of the critical behavior of the compact case, which we have argued in the introduction to be the same as for the non-compact Abelian Higgs model at infinite *bare* gauge coupling. Before doing this, however, a few preliminary remarks are in order.

The Abelian Higgs model is manifestly a two-scale theory. Indeed, the gauge field becomes massive due to the Higgs mechanism. Thus, in the ordered phase we are left with two mass scales, the Higgs mass m and the gauge field mass m_A. From these two mass scales we obtain the Ginzburg parameter $\kappa \equiv m/m_A$. Due to the existence of two mass scales in the problem, we have very distinct situations depending on whether $\kappa \ll 1$ or $\kappa \gg 1$. For $\kappa \ll 1$ vortex lines, which are the topological defects of the matter field, attract each other. This corresponds to a type I regime, while for $\kappa \gg 1$ we have repulsive forces between vortex lines, which corresponds to the type II regime. This two-scale behavior survives in the disordered phase, though in this case $m_A = 0$.

We shall consider the calculation of RG functions for the massless theory, but using two renormalization scales [21]. In order to see the influence of the two mass scales appearing in the ordered phase, on the massless theory, we define the dimensionful couplings at different renormalization points, u at μ and e^2 at $\bar{\mu}$. Let us define the ratio $r = \mu/\bar{\mu}$. By rewriting $e^2(\bar{\mu})$ in terms of μ, we obtain the one-loop β-functions for any fixed dimension $d \in (2, 4]$ and an order parameter with $N/2$ complex components [50]

$$\beta_\alpha = (4-d)\left[-\alpha + rNA(d)\alpha^2\right], \tag{86}$$

$$\beta_g = (4-d)\left\{-g + B(d)\left[-2(d-1)\alpha g + \frac{N+8}{2}g^2 + 2(d-1)\alpha^2\right]\right\}, \tag{87}$$

where

$$A(d) = -\frac{\Gamma(1-d/2)\Gamma^2(d/2)}{(4\pi)^{d/2}\Gamma(d)}, \tag{88}$$

$$B(d) = \frac{\Gamma(2-d/2)\Gamma^2(d/2-1)}{(4\pi)^{d/2}\Gamma(d-2)}. \tag{89}$$

From Eq. (86) we see that $\gamma_A = r(4-d)NA(d)\alpha$. By considering $d = 4-\epsilon$ and expanding for small ϵ, we recover the well-known ϵ-expansion result [51] if we take $r = 1$. In our fixed dimension approach r is an arbitrary parameter that is usually fixed by imposing

[1] In an early Monte Carlo simulation, a tricritical value $\kappa_{\text{tri}} = 0.4/\sqrt{2}$ was found, [48]. This is the value used in the ad hoc scheme of Ref. [21]. More recently, a large-scale Monte Carlo simulation improved on this value, finding $\kappa_{\text{tri}} = (0.76 \pm 0.04)/\sqrt{2}$, [49]. This is in surprisingly good agreement with an early analytical result $\kappa_{\text{tri}} = 0.798/\sqrt{2}$, see Ref. [27]. Using this improved value for κ_{tri} in the β-functions of Ref. [21], the critical exponent ν obtained would be $\nu = 0.53$. This is quite far from the correct $3DXY$ value $\nu_{XY} = 0.67$, as well as from the $3DXY$ one-loop value $\nu = 0.625$.

additional conditions [21]. When $d = 3$ and $N = 2$ we have the fixed point $\alpha_*(r) = 16/r$. In the context of the compact Abelian Higgs model we fix the value of r by demanding that K_c should correspond to a $r = r_c$, with $K = 1/\alpha_*$ at one-loop. If we use the spin-wave estimate $K_c = 3$ (which corresponds to $t_c = 12\pi^2$), we obtain then that $r_c = 48$ and thus $\alpha_* = 1/3$. On the other hand, if we use the estimate from our KT-like recursion relations, we have $K_c = 5/2$ and therefore $\alpha_* = 2/5$. In order to check the quality of these matchings, we compute the critical exponents of the three-dimensional Abelian Higgs model in $d = 3$. The critical exponent ν is given by the fixed point value of the RG function

$$\nu_\phi = \frac{1}{2 + \gamma_m}, \tag{90}$$

where

$$\gamma_m = \mu \frac{\partial \ln Z_m}{\partial \mu} - \gamma_\phi, \tag{91}$$

with Z_m being the mass renormalization and

$$\gamma_\phi = \mu \frac{\partial \ln Z_\phi}{\partial \mu}. \tag{92}$$

At the fixed point γ_ϕ gives the value of the critical exponent η. At one-loop order, we have

$$\gamma_m = \frac{\alpha - g}{4}, \qquad \gamma_\phi = -\frac{\alpha}{4}. \tag{93}$$

When $K_c = 3$, the fixed point for the coupling g which corresponds to infrared stability is given by $g_* = 2(7 + 2\sqrt{11})/15$. Therefore, we obtain $\nu \approx 0.615$ and $\eta = -1/12$. Using $K_c = 5/2$, we obtain $g_* = 4(6 + \sqrt{31})/25$. The critical exponents in this case are $\nu \approx 0.61$ and $\eta = -1/10$. Both estimates are close to the one-loop value of the XY model, $\nu_{XY} \approx 0.625$. From duality arguments we expect indeed a XY value for the exponent ν [52].

6. Summary and discussion

In this paper, we have considered the Abelian Higgs model in $2 + 1$ dimensions both for the non-compact and compact cases, with matter fields in the fundamental representation. We have performed a duality lattice transformation on these models, emphasizing the features that set them apart as well as those they have in common. A major difference lies in the fact that in the dual formulation, the non-compact case has stringent constraints $\nabla \cdot \mathbf{l}_i = 0$ imposed on the topological currents of the system, while in the compact case $\nabla \cdot \mathbf{l}_i$ can take any integer value, i.e., the currents are unconstrained *for the case where the matter field is in the fundamental representation*. This effectively makes the dual non-compact case a much more strongly interacting system of topological currents, and this is why phase transitions are more easily brought out compared to the compact case. As a result, we have seen that there is one limit of the LAH model where the non-compact case exhibits the IXY transition, while the compact case is an exactly soluble discrete Gaussian model with apparently no phase transition.

A major part of the paper (Sections 3 and 4) has been devoted to establishing that, despite the absence of any phase transitions with a local order parameter in the compact case, a topological phase transition nevertheless is found in the interior of the phase diagram of the model. A key ingredient is the renormalization of the gauge-field propagator of the problem due to critical matter field fluctuations, Eq. (46). With no matter fields present, the topological defects of the gauge field, which are monopole configurations, interact with a $1/R$ potential in $d = 3$. In the presence of matter fields, taking into account their critical fluctuations, the resultant effective gauge theory may be described as an overall neutral plasma of charges that interact with a logarithmic potential in $d = 3$, Eq. (49). A field-theoretical formulation of the action given in Eq. (49) yields an *anomalous* sine-Gordon (ASG) model, Eq. (51). A renormalization group analysis of this model based on the Callan–Symanzik equations shows that the theory is massive below a critical value of the coupling constant. This by itself suffices to conclude that a phase transition exists. We then go on to show that the problem is amenable to an analysis based on KT-like recursion relations, Eqs. (82), derived for a d-dimensional gas of point charges interacting with a pair-potential which in a certain limit is logarithmic. In this limit, the recursion relations we derive for the stiffness and fugacity of the problem reduce to equations which are similar in structure to the well-known Kosterlitz–Thouless recursion relations obtained for the two-dimensional Coulomb gas, but with a modified equation for the fugacity due to an induced anomalous scaling of it. This anomalous scaling in the fugacity accounts for deviations from the ordinary Coulomb gas case in d dimensions. The change in the equation for the fugacity shows that the stiffness and the fugacity of the problem mutually influence each other under renormalization in a manner which is different from the case of a logarithmic pair-interaction in $d = 2$. As a consequence of this, the universal jump in the stiffness at the transition is then given, in appropriate units, by the dimensionality of the system and the anomalous scaling of the fugacity, Eq. (85).

In Section 5, we have seen that the deconfinement phase transition we find in the compact case, with a critical coupling t_c, allows us to fix a parameter appearing in the evaluation of the critical exponents of the non-compact Abelian Higgs model. This represents an improvement on previous schemes to fix this parameter.

We close with a few remarks on unsolved problems. When only fermionic fields are coupled to the massless gauge field (spinor QED_3), then we again obtain a β-function for the renormalized gauge coupling as given in Eq. (4), but γ_A in the equation now depends only on one coupling constant, α, not two as in the bosonic case. Then we do not have the freedom to tune parameters of the model to drive it through a phase transition of the type described in Section 4. The analysis of Section 4 may be carried through as before, but the point is that the fixed point coupling, $\alpha = \alpha_*$ does not depend on any second coupling constant g, this simply does not appear in the theory. Instead, α_* depends on the number of fermion flavours N only. In principle there thus exists a critical value $N = N_c$ where the compact version of the model with fermionic matter, also goes through a deconfinement transition. The confining phase corresponds to $N < N_c$. It is highly controversial what this critical value is. A simple one-loop renormalization group calculation gives $N_c = 24$ [20] in agreement with an earlier result by Ioffe and Larkin obtained by a quite different method [31]. However, we may in fact expect that the actual value is much smaller than this. Marston has calculated the same number using one-instanton action and finds $N_c = 0.9$

[53]. The important point here is that whatever the precise value of N_c is, the interaction between the monopoles is always logarithmic.

Also, in the fermionic case there is a subtlety in that another type of instability, absent in the bosonic case, could intervene to destroy the deconfinement transition. Fermions can in principle undergo a spontaneous chiral symmetry breaking (SχSB) [54]. This happens when the number of fermion flavours is less than some critical value, N_{ch} say. This means that a fermion mass is dynamically generated for $N < N_{\text{ch}}$. The precise value of N_{ch} is presently also a matter of debate. One estimate from the Schwinger–Dyson equation gives $N_{\text{ch}} = 32/\pi^2$ [55]. This result is confirmed by Monte Carlo simulations finding $N_{\text{ch}} \approx 3.5$ [56]. Another analytic calculation gives $N_{\text{ch}} = 128/3\pi^2$ [57]. A recent estimate based on a new constraint on strongly interacting systems gives $N_{\text{ch}} \leqslant 3/2$ [58]. This is quite consistent with the most recent numerical results we are aware of [59], where no signs of SχSB is found for $N \geqslant 2$. Thus, there is no consensus on the precise value of N_{ch}. The calculation of N_c assumes that the fermions are massless. Thus, if $N_c = 24$ as in Refs. [20,31], then a deconfinement transition will take place since the fermion mass is generated at a much lower value of N. With massive fermions present our anomalous three-dimensional compact QED scenario does not apply because the Maxwell term does not become irrelevant anymore. In such a situation the results of Polyakov [8] apply and there is permanent confinement of electric test charges. This would be the case for the value $N_c = 0.9$ obtained by Marston [53], which lies below all estimates of N_{ch}. In this case the deconfinement transition does not happen.

Physically, SχSB in spinor QED$_3$ has important consequences in the physics of high-T_c cuprates. As we mentioned in the introduction, spinor QED$_3$ with a compact gauge field emerges as a possible low energy description of the fluctuations around the flux phase in the quantum Heisenberg antiferromagnet [1]. In this context, the dynamical mass generation is associated with the spin density wave (SDW) instability. Thus, gauge field fluctuations could in principle restore the Néel state. The physical number of fermion components in this case is $N = 2$. Spinor QED$_3$ also emerges by considering the low energy physics of the d-wave superconducting state in the pseudogap phase of the high-T_c cuprates [60]. In this case, however, the gauge field is non-compact and there is an inherent anisotropy in the Lagrangian. There also, SχSB is responsible for the onset of SDW as half-filling is approached [61]. The physical number of fermion components in this case is again $N = 2$. Therefore, in these theories it is essential that $N_{\text{ch}} > 2$. If the most recent estimate for N_{ch} is correct [58], this could have serious implications for the validity of the different spinor QED$_3$ scenarios discussed above. In the case of the spinor QED$_3$ description of the pseudogap phase, the inherent anisotropy could possibly affect the value of N_{ch}. However, results presented thus far indicate that at least weak anisotropy will not affect N_{ch} obtained in the isotropic case [62]. Moreover, when studying effective theories of undoped high-T_c cuprates, we have argued in the introduction that the relevant theory to study is fermions coupled to compact $U(1)$ gauge-fields. Hence, it is of importance to revisit the problem of how monopoles affects SχSB [63]. Finally, we note that a recent provocative paper by Wen [64] states that there exists a principle of *quantum order* which may prevent fermions from dynamically acquiring a mass even in the presence of strong coupling to gauge fields. Hence, it seems to us that a renewed effort in numerical computations of N_{ch}

in $(2+1)$-dimensional gauge theories coupled to fermionic matter, including the effects of compactness and anisotropy, would be very timely.

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Appendix A. KT-like transition in three dimensions in an anisotropic sine-Gordon theory

While considering a class of globally symmetric self-dual Z_N models in the $N \to \infty$ limit, Amit et al. [46] arrived at the following anisotropic three-dimensional sine-Gordon action containing higher derivatives:

$$S_{\text{ANISG}} = \int d^3x \left[\frac{1}{2t_1} \left(\partial_\parallel^2 \varphi\right)^2 + \frac{1}{2t_2} (\partial_z \varphi)^2 - z \cos\varphi \right], \tag{A.1}$$

where $\partial_\parallel^2 = \partial_x^2 + \partial_y^2$. As pointed out in Ref. [46], the above model has a KT transition in three dimensions. Indeed, it is easy to see that the propagator is logarithmic at large distances. Note, however, that anisotropy and the higher order derivatives in the parallel direction are essential, and the system effectively shows two-dimensional behavior by dimensional reduction. This is in contrast with our genuinely three-dimensional KT-like scenario.

Appendix B. KT-like recursion relations

In this appendix we derive to lowest order in the fugacity the recursion relations for the scale-dependent stiffness parameter $K(l)$ and fugacity $y(l)$ given in Eqs. (82) for a d-dimensional plasma where the bare pair-potential is given by Eq. (81), which reduces to a logarithmic potential when $d = 3$. The starting point will be a low-density approximation for a dielectric constant of this system. We closely follow a method for doing this introduced in [65]. Introducing the solid angle in d dimensions $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ and the density of dipoles in the fluid by n_d, a low-density approximation for the dielectric constant is given by

$$\varepsilon = 1 + n_d \Omega_d \alpha, \tag{B.1}$$

where α here denotes the polarizability of the medium, a standard linear-response analysis gives $\alpha = 4\pi^2 K \langle s^2 \rangle / d$ and $\langle s^2 \rangle$ is the mean square of the dipole moment in the system. To compute this, we need the low-density limit of the pair-distribution function $n^{\pm}(r)$ of the plasma, which is readily obtained from the grand canonical partition function Ξ expanded to second order in the bare fugacity ζ, and replacing the thermal de Broglie wavelength by a short-distance cutoff r_0, as follows

$$n^{\pm}(r) = \frac{\zeta^2}{r_0^{2d}} e^{-4\pi^2 K V}. \tag{B.2}$$

In this way, we may now go on to express a *scale-dependent* dielectric constant as follows

$$\varepsilon(r) = 1 + \frac{4\pi^2 \Omega_d K}{d} \int_{r_0}^{r} ds\, s^{d+1} n^{\pm}(s). \tag{B.3}$$

Note however, that in Eq. (B.3), a mean-field approximation is understood to be used by replacing the bare potential V in $n^{\pm}(r)$ by an *effective potential* $U(r)$. This effective screened potential must be selfconsistently determined by demanding that it gives rise to an electric field in the problem given by

$$\frac{\partial U}{\partial r} = E(r) = \frac{\tilde{f}(d)}{\varepsilon(r) r^{1-\rho}}, \tag{B.4}$$

where $\rho = 2 - d + \eta_A$ and $\tilde{f}(d)$ is defined in Eq. (84). Such a mean-field procedure has been consistently used with success in the 2d case, and the origin of the success lies in the long range of the ln-interaction. In higher dimensions, such a procedure will work even better since the logarithmic potential is felt over even longer distances due to extra volume factors.

Let us introduce a logarithmic length scale $l = \ln(r/r_0)$ along with the new variables

$$\tau(l) = \frac{\varepsilon(r_0 \exp l)}{4\pi^2 K},$$
$$x(l) = 4\pi^2 K U(r_0 \exp l). \tag{B.5}$$

Here, $x(l)$ is determined selfconsistently by integrating the effective field $E(r)$. Then we get from Eqs. (B.3) and (B.4)

$$\tau(l) = \tau(0) + \frac{\Omega_d \zeta^2}{d r_0^{d-2}} \int_0^l dv\, e^{(d+2)v - x(v)}, \tag{B.6}$$

and

$$x(l) = x(0) + \tilde{f}(d) \int_0^l dv\, \frac{r_0^{\rho} e^{\rho v}}{\tau(v)}. \tag{B.7}$$

From Eqs. (B.6) and (B.7), we may derive coupled renormalization group equations for $\tau(l)$ and $x(l)$. However, in order to obtain equations that have a form more similar to equations that have appeared in the literature on the d-dimensional Coulomb gas [44], we introduce a new variable $K(l)$ representing a scale dependent stiffness constant, as follows

$$K^{-1}(l) \equiv \frac{\tau(l)}{r_0^{\rho} e^{\rho l}}. \tag{B.8}$$

Thus, we see that the effect of a nonzero ρ on the stiffness amounts to a scaling change $K(l) \to e^{\rho l} K(l)$. Using Eq. (B.7), we have that

$$\frac{\partial x(l)}{\partial l} = 4\pi^2 \tilde{f}(d) K(l). \tag{B.9}$$

Differentiating $K^{-1}(l)$ with respect to l and using Eq. (B.6), we obtain

$$\frac{\partial K^{-1}(l)}{\partial l} = -\rho K^{-1}(l) + \frac{2\Omega_d \zeta^2}{d r_0^{d-2+\rho}} e^{[(d+2-\rho)l - x(l)]}. \tag{B.10}$$

From this expression, we define a scale dependent fugacity $y(l)$ given by

$$y(l) \equiv \frac{\sqrt{2\Omega_d}\,\zeta\, e^{[(d+2-\rho)l - x(l)]/2}}{\sqrt{d}\, r_0^{(d-2+\rho)/2}}. \tag{B.11}$$

Thus, we see explicitly that the renormalization of $K(l)$ in principle influences the flow equation for $y(l)$, which is obtained by differentiating with respect to l and using Eq. (B.9)

$$\frac{\partial y(l)}{\partial l} = \left[d - \eta_y - 2\pi^2 \tilde{f}(d) K(l)\right] y(l), \tag{B.12}$$

where $\eta_y = (d - 2 + \rho)/2$. Eqs. (B.10) and (B.12) are precisely Eqs. (82). On the other hand, the Callan–Symanzik approach of Section 4.1, which basically ignores the influence of *the renormalization of* $K(l)$ on the structure of the flow equation for $y(l)$, yields as we have seen Eq. (77). We have already remarked in Section 4.2 that this type of approach gives the correct answer only if there are no deviations from the Coulomb potential case, that is, we need $\rho = 2 - d$. Note that in the usual KT transition we would have $\rho = \eta_y = 0$.

Appendix C. Screened effective potential

In this appendix, we derive the asymptotic long-distance behavior of the screened effective interaction $U(r)$ introduced in Appendix B, for the case $\rho = 0$, corresponding to $d = 3$ and $\eta_A = 1$. We start from the recursion relations, written on the form

$$\frac{\partial K^{-1}}{\partial l} = y^2, \qquad \frac{\partial y}{\partial l} = \left[\frac{5}{2} - K(l)\right] y. \tag{C.1}$$

From Eq. (B.8), we have that $K^{-1}(l) = \tau(l)$ in this case. Next, we introduce the variable $T(l)$ defined by

$$T(l) \equiv \frac{5\tau(l)/2 - 1}{5\tau(l)/2} \approx \frac{5}{2}\tau(l) - 1, \tag{C.2}$$

where the latter approximation is asymptotically exact close enough to the transition. In terms of this, the flow equation for the fugacity may be written on the form

$$\frac{\partial y^2(l)}{\partial l} = 5T(l)y^2(l). \tag{C.3}$$

On the other hand, we have

$$\frac{\partial T^2(l)}{\partial l} \approx 5T(l)\frac{\partial \tau(l)}{\partial l} = 5T(l)y^2(l), \tag{C.4}$$

and hence we have

$$y^2(l) - T^2(l) = \pm\omega^2, \tag{C.5}$$

where ω is some positive number. We are interested in the quantity $\lim_{l\to\infty} x(l)$ for the case where $y^2(l) - T^2(l) < 0$, and $T(l) < 0$, this will be the regime where the fugacity scales to zero. In this case we choose the negative sign on the r.h.s. in Eq. (C.5). From the flow equation for $K^{-1}(l)$ we find

$$\frac{\partial T(l)}{\partial l} = \frac{5}{2}y^2(l) = -\frac{5}{2}\left[\omega^2 - T^2(l)\right]. \tag{C.6}$$

This is solved to obtain, introducing $u = (5/2)\omega l + \theta$,

$$\begin{aligned} T(l) &= -\omega \coth u, \\ y(l) &= \frac{\omega}{\sinh u}, \end{aligned} \tag{C.7}$$

where ω and θ are integration constants that are uniquely determined from the initial conditions on $\tau(l)$ and $y(l)$, i.e., by the bare coupling constants of the problem as follows

$$\begin{aligned} y^2(0) - T^2(0) &= -\omega^2, \\ \frac{T(0)}{y(0)} &= -\cosh\theta. \end{aligned} \tag{C.8}$$

From the expression for $T(l)$, using Eq. (C.2), we obtain

$$\tau(l) = \frac{2}{5}(1 - \omega\coth u). \tag{C.9}$$

Since $\tau(l) > 0$, this puts restrictions on the constants ω and θ, and the most severe limitations on ω in terms of θ is given by

$$1 - \omega\coth\theta > 0. \tag{C.10}$$

Using Eq. (B.9) and the fact that $K(l) = 1/\tau(l)$, we have

$$\frac{\partial x(l)}{\partial l} = \frac{5/2}{1 - \omega\coth u}. \tag{C.11}$$

From Eq. (C.10), we see that $\partial x(l)/\partial l > 0$. This is an important result, since it immediately reveals that, in the regime $y^2(l) - T^2(l) < 0$ we consider here, the logarithmic bare potential $V(r)$ cannot possibly be screened into a power law potential $1/r^\sigma$ with $\sigma > 0$, since in that case we would have $\partial x(l)/\partial l < 0$. However, for all l we have

$$\frac{\partial^2 x(l)}{\partial l^2} = -\left(\frac{5\omega/2}{\sinh u - \omega \cosh u}\right)^2 < 0. \tag{C.12}$$

Introducing $\omega_\pm = 1 \pm \omega$, Eq. (C.11) is straightforwardly integrated to yield

$$x(l) - x(0) = \frac{1}{\omega_+\omega_-}\left[\frac{5}{2}\omega_+ l + \ln\left(\frac{\omega_+ e^{-2\theta} + \omega_-}{\omega_+ e^{-2u} + \omega_-}\right)\right]. \tag{C.13}$$

From this, it follows that for $r \gg r_0$ the effective potential behaves asymptotically as

$$U(r) \sim \ln(r/r_0). \tag{C.14}$$

Appendix D. Exact equation of state for the d-dimensional ln-plasma

The equation of state for a d-dimensional ln-plasma with no short-distance cutoff, may be obtained via a simple scaling argument, previously applied to the two-dimensional case [66]. The configurational integral in the canonical partition function is given by

$$Q = \int_V \cdots \int_V d^d\mathbf{r}_1 \cdots d^d\mathbf{r}_{2N} \exp\left[\tilde{t} \sum_{i<j} q_i q_j \ln(r_{ij})\right], \tag{D.1}$$

where $q_i = \pm 1$, and we assumed that we have $2N$ particles in the system, N with charge $q_i = 1$ and N with charge $q_i = -1$, $\sum_{i=1}^{2N} q_i = 0$. Here, $V = L^d$ is the volume of the system. Introduce new dimensionless variables $R_{ij} = r_{ij}/L$ where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$, in which case the configurational integral is given by

$$\begin{aligned} Q &= L^{2Nd} \int_0^1 \cdots \int_0^1 d^d R_1 \cdots d^d R_{2N} \exp\left(\tilde{t} \sum_{i<j} q_i q_j \ln(R_{ij} L)\right) \\ &= L^{2Nd} \exp\left[\tilde{t} \sum_{i<j} q_i q_j \ln(L)\right] I, \end{aligned} \tag{D.2}$$

where the integral I is independent of volume. Now note that

$$2\sum_{i<j} q_i q_j = \sum_{i \neq j} q_i q_j = \left(\sum_i q_i\right)\left(\sum_j q_j\right) - \sum_{i=1}^{2N} q_i^2 = -2N. \tag{D.3}$$

Then we obtain

$$Q = L^{2Nd} e^{-\tilde{t}N \ln(L)} I = L^{2Nd - \tilde{t}N} I = V^{2N - \tilde{t}N/d} I. \tag{D.4}$$

From this, we obtain the equation of state involving the pressure

$$\tilde{t} p V = 2N - \frac{\tilde{t}N}{d}. \tag{D.5}$$

Note that the pressure vanishes when $\tilde{t} = \tilde{t}_0 = 2d$. A prerequisite for the validity of the above analysis is that the quantity I must be finite, otherwise the scaling of variables that lead to the equation of state is meaningless. In fact, I is not always finite. Consider again the integrand in Q, which is given by a product of factors

$$e^{\tilde{t}\sum_{i<j} q_i q_j \ln(r_{ij})} = \prod_{i<j} r_{ij}^{\tilde{t} q_i q_j}. \tag{D.6}$$

Any factor with $q_i = -q_j$ will be singular when $r_{ij} = 0$, which is possible in the absence of a short-distance cutoff. To investigate whether or not this singularity is integrable, consider the integral

$$\int dr\, r^{d-1} r^{-\tilde{t}}. \tag{D.7}$$

This is finite only if

$$d - \tilde{t} > 0. \tag{D.8}$$

This means that the equation of state Eq. (D.5) makes sense for $\tilde{t} < \tilde{t}_c = d$, note that for all dimensions d, $\tilde{t}_0 = 2\tilde{t}_c$.

In two dimensions, it is known that the negativity of the pressure occurs at a temperature that coincides with the KT vortex–antivortex unbinding temperature, and that there is a phase transition at twice this temperature. It is amusing to note here that in the three-dimensional case, the pressure vanishes at $t_c = 12\pi^2$, after having reintroduced $\tilde{t} = t/4\pi^2$. This is precisely the critical coupling we found in Section 4.1 from the Callan–Symanzik equations. In addition there is again a phase transition at precisely half the value of this coupling constant, where the pressure becomes that of an ideal gas of N particles. In arbitrary dimensions, this persists, the phase transition to an ideal gas of N particles always happens at half of the value at which the pressure vanishes. This phase transition, which is a collapse of an overall charge-neutral plasma of N $q_i = +1$ charges and N $q_i = -1$ charges into an ideal gas of N particles, occurs because of the lack of a short-distance cutoff in the system we consider in this appendix.

Appendix E. Duality in the Abelian compact Higgs model with a Chern–Simons term

For completeness, we present in this appendix the duality transformation of the LAH with a Chern–Simons term added [12]. Compact gauge theories with Chern–Simons term added are relevant in studies of chiral spin liquid states [11] when spinor states have been integrated out. Such theories have been argued to exhibit a deconfinement transition [67, 68]. The compact LAH mode, i.e., $A_{i\mu} \in (-\pi, \pi)$, with a Chern–Simons term has the action

$$\begin{aligned} S_{\mathrm{CS}} = \sum_i \bigg[&\frac{\beta}{2}(\nabla_\mu \theta_i - A_{i\mu} - 2\pi n_{i\mu})^2 + \frac{1}{2e^2}(\epsilon_{\mu\nu\lambda}\nabla_\nu A_{i\lambda} - 2\pi N_{i\mu})^2 \\ &+ i\frac{\gamma}{2}(\nabla_\mu \theta_i - A_{i\mu} - 2\pi n_{i\mu})(\epsilon_{\mu\nu\lambda}\nabla_\nu A_{i\lambda} - 2\pi N_{i\mu}) \bigg]. \end{aligned} \tag{E.1}$$

Let us introduce auxiliary fields $\mathbf{a}_i$, $\mathbf{b}_i$, $\lambda_{i\mu}$, and $\sigma_{i\mu}$, such that

$$S'_{\mathrm{CS}} = \sum_i \biggl[\frac{\beta}{2}\mathbf{a}_i^2 + \frac{1}{2e^2}\mathbf{b}_i^2 + i\frac{\gamma}{2}\mathbf{a}_i \cdot \mathbf{b}_i + i\lambda_{i\mu}(\nabla_\mu\theta_i - A_{i\mu} - 2\pi n_{i\mu} - a_{i\mu}) + i\sigma_{i\mu}(\epsilon_{\mu\nu\lambda}\nabla_\nu A_{i\lambda} - 2\pi N_{i\mu} - b_{i\mu}) \biggr]. \tag{E.2}$$

Next we introduce integer valued fields $m_{i\mu}$ and $M_{i\mu}$ via the Poisson formula:

$$S''_{\mathrm{CS}} = \sum_i \biggl[\frac{\beta}{2}\mathbf{a}_i^2 + \frac{1}{2e^2}\mathbf{b}_i^2 + i\frac{\gamma}{2}\mathbf{a}_i \cdot \mathbf{b}_i + im_{i\mu}(\nabla_\mu\theta_i - A_{i\mu} - a_{i\mu}) + iM_{i\mu}(\epsilon_{\mu\nu\lambda}\nabla_\nu A_{i\lambda} - b_{i\mu}) \biggr]. \tag{E.3}$$

Integration of θ_i and $A_{i\mu}$ give the constraints enforced by delta of Kronecker

$$\nabla \cdot \mathbf{m}_i = 0, \tag{E.4}$$

$$\nabla \times \mathbf{M}_i = \mathbf{m}_i. \tag{E.5}$$

Summing over $\mathbf{m}_i$ gives

$$S'''_{\mathrm{CS}} = \sum_i \biggl[\frac{\beta}{2}\mathbf{a}_i^2 + \frac{1}{2e^2}\mathbf{b}_i^2 + i\frac{\gamma}{2}\mathbf{a}_i \cdot \mathbf{b}_i - i(\nabla \times \mathbf{M}_i)\cdot\mathbf{a}_i - i\mathbf{M}_i \cdot \mathbf{b}_i \biggr]. \tag{E.6}$$

By integrating out $\mathbf{a}_i$ and $\mathbf{b}_i$ we arrive at the action

$$\widetilde{S}_{\mathrm{CS}} = \frac{K}{2}\sum_i \bigl[(\nabla \times \mathbf{M}_i)^2 + \beta e^2\mathbf{M}_i^2 - ie^2\gamma\mathbf{M}_i \cdot (\nabla \times \mathbf{M}_i)\bigr], \tag{E.7}$$

where $K \equiv 4/(\gamma^2 e^2 + 4\beta)$. Using the Poisson formula to introduce a real lattice field $h_{i\mu}$ and doing an appropriate rescaling of the variables we obtain finally the partition function

$$Z = Z_0 \sum_{\{\mathbf{l}_i\}} \int_{-\infty}^{\infty} \biggl[\prod_{i,\mu} dh_{i\mu}\biggr] \exp\bigl[-S_{\mathrm{CS}}^{\mathrm{dual}}(\mathbf{h}_i, \mathbf{l}_i)\bigr], \tag{E.8}$$

where

$$S_{\mathrm{CS}}^{\mathrm{dual}} = \frac{K}{2}\sum_i \bigl[(\nabla \times \mathbf{h}_i)^2 + \beta e^2\mathbf{h}^2 - i\gamma e^2\mathbf{h}_i \cdot (\nabla \times \mathbf{h}_i)\bigr] + i2\pi\mathbf{l}_i \cdot \mathbf{h}_i, \tag{E.9}$$

which should be compared with Eqs. (26) and (34). Note the appearance of the cross-term $i\gamma e^2\mathbf{h}_i \cdot (\nabla \times \mathbf{h}_i)$. When the $\mathbf{h}_i$ are integrated out we are thus left with a partition of the same form as Eq. (35), but with an asymmetric propagator.

If we were to consider the non-compact LAH with a Chern–Simons term added, and in the absence of the Maxwell term, $e^2 \to \infty$, then this is an effective description of the fractional quantum Hall effect [37,69]. In this case we obtain

$$S_{\mathrm{CS}}^{\mathrm{dual}} = \sum_i \biggl[\frac{1}{2\beta}(\nabla \times \mathbf{h}_i)^2 - \frac{i}{2\gamma}\mathbf{h}_i \cdot (\nabla \times \mathbf{h}_i)\biggr] + i2\pi\mathbf{l}_i \cdot \mathbf{h}_i. \tag{E.10}$$

This is essentially the same as Eqs. (E.8) and (E.9) for the compact case (with no mass term for the $\mathbf{h}_i$-fields), but we should add an additional constraint in the $\nabla \cdot \mathbf{l}_i = 0$ in the partition function.

One point worth emphasizing here, sometimes overlooked, is that the gauge-field $\mathbf{h}_i$ is never a compact gauge-field, whether one starts from an original compact or non-compact gauge theory. In the non-compact Chern–Simons theory, there exists a self-dual point at a value $\gamma = 1/2\pi$ [69,70]. The possibility of self-duality is a consequence of non-compactness, it can never arise starting from a compact LAH model with Chern–Simons term added. It is an intriguing question whether the self-duality at the above particular value of γ in the non-compact case corresponds to a critical point. A candidate physical interpretation of such a putative phase transition would correspond to statistical transmutation of the Laughlin quasiparticles of the fractional quantum Hall effect as magnetic field is varied, since in the context of the FQHE, the parameter γ depends on filling fraction, i.e., magnetic field. It is known that for the half-filled lowest Landau level, the quasiparticles are fermions [71], while for other filling fractions they are anyons.

References

[1] I. Affleck, J.B. Marston, Phys. Rev. B 37 (1988) 627;
J.B. Marston, I. Affleck, Phys. Rev. B 39 (1989) 11538.
[2] G. Baskaran, P.W. Anderson, Phys. Rev. B 37 (1988) 580.
[3] I. Affleck, Z. Zou, T. Hsu, P.W. Anderson, Phys. Rev. B 38 (1988) 745;
E. Dagotto, E. Fradkin, A. Moreo, Phys. Rev. B 38 (1988) 2926.
[4] M. Einhorn, R. Savit, Phys. Rev. D 19 (1979) 1198.
[5] E. Fradkin, S.H. Shenker, Phys. Rev. D 19 (1979) 3682.
[6] M.N. Chernodub, E.-M. Ilgenfritz, A. Schiller, hep-lat/0207020;
M.N. Chernodub, E.-M. Ilgenfritz, A. Schiller, hep-lat/0208013.
[7] T. Matsui, in: W. Jahnke, et al. (Eds.), Fluctuating Paths Fields, World Scientific, 2001, pp. 271–280, cond-mat/0112463;
M. Kemuriyama, T. Matsui, cond-mat/0203136;
Y. Fujita, T. Matsui, cond-mat/0207023.
[8] A.M. Polyakov, Nucl. Phys. B 120 (1977) 429.
[9] C. Nayak, Phys. Rev. Lett. 85 (2000) 178.
[10] C. Mudry, E. Fradkin, Phys. Rev. B 50 (1994) 11409.
[11] V. Kalmeyer, R.B. Laughlin, Phys. Rev. Lett. 59 (1988) 2095;
X.G. Wen, F. Wilczeck, A. Zee, Phys. Rev. B 39 (1989) 11413.
[12] S. Deser, R. Jackiw, S. Templeton, Phys. Rev. Lett. 48 (1982) 975;
S. Deser, R. Jackiw, S. Templeton, Ann. Phys. (N.Y.) 140 (1982) 372.
[13] T. Senthil, M.P.A. Fisher, Phys. Rev. B 62 (2000) 7850.
[14] N. Read, S. Sachdev, Phys. Rev. Lett. 66 (1991) 1773;
S. Sachdev, N. Read, Int. J. Mod. Phys. B 5 (1991) 219.
[15] X.-G. Wen, Phys. Rev. B 44 (1991) 2664.
[16] T. Senthil, M.P.A. Fisher, J. Phys. A 34 (2001) L119.
[17] N. Nagaosa, P.A. Lee, Phys. Rev. B 61 (2000) 9166.
[18] I. Ichinose, T. Matsui, Phys. Rev. Lett. 86 (2001) 942;
I. Ichinose, T. Matsui, M. Onoda, Phys. Rev. B 64 (2001) 104516.
[19] C. Nayak, Phys. Rev. Lett. 86 (2001) 943.
[20] H. Kleinert, F.S. Nogueira, A. Sudbø, Phys. Rev. Lett. 88 (2002) 232001.
[21] I.F. Herbut, Z. Tešanović, Phys. Rev. Lett. 76 (1996) 4588.

[22] J. Hove, A. Sudbø, Phys. Rev. Lett. 84 (2000) 3426.
[23] A. Kovner, P. Kurzepa, B. Rosenstein, Mod. Phys. Lett. A 8 (1993) 1343.
[24] A.K. Nguyen, A. Sudbø, Phys. Rev. B 60 (1999) 15307;
J. Hove, S. Mo, A. Sudbø, Phys. Rev. Lett. 85 (2000) 2368.
[25] P. Olsson, S. Teitel, Phys. Rev. Lett. 80 (1998) 1964.
[26] C. Dasgupta, B.I. Halperin, Phys. Rev. Lett. 47 (1981) 1556.
[27] H. Kleinert, Lett. Nuovo Cimento 35 (1982) 405;
H. Kleinert, Gauge Fields in Condensed Matter, Vol. 1, World Scientific, Singapore, 1989, http://www.physik.fu-berlin.de/~kleinert/re.html#b1.
[28] H. Kleinert, V. Schulte-Frohlinde, Critical Phenomena in Φ^4-Theory, World Scientific, Singapore, 2001, http://www.physik.fu-berlin.de/~kleinert/b8.
[29] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, Oxford Univ. Press, Oxford, 1993.
[30] D.H. Kim, P.A. Lee, Ann. Phys. (N.Y.) 272 (1999) 130.
[31] L.B. Ioffe, A.I. Larkin, Phys. Rev. B 39 (1989) 8988.
[32] S. Sachdev, K. Park, Ann. Phys. (N.Y.) 298 (2002) 58.
[33] G. 't Hooft, in: A. Zichichi (Ed.), High Energy Physics: Proceedings of the EPS International Conference, Palermo, June 1975, Editrici Compositori, Bologna, 1976;
S. Mandelstam, in: J.L. Gervais, A. Neveu (Eds.), Extended Systems in Field Theory, Phys. Rep. 23 (1976) 245.
[34] H. Kleinert, Phys. Lett. B 293 (1992) 168;
H. Kleinert, Phys. Lett. B 246 (1990) 127.
[35] H. Kleinert, Theory of fluctuating nonholonomic fields and applications: statistical mechanics of vortices and defects and new physical laws in spaces with curvature and torsion, in: A.-C. Davis, et al. (Eds.), Formation and Interactions of Topological Defects, in: NATO Asi Series B, Physics, Vol. 349, Plenum, New York, 1995.
[36] R.E. Peierls, Ann. Inst. H. Poincaré 5 (1935) 177;
N.D. Mermin, H. Wagner, Phys. Rev. Lett. 22 (1966) 1133;
P.C. Hohenberg, Phys. Rev. 158 (1967) 383;
N.D. Mermin, Phys. Rev. 176 (1968) 250.
[37] E. Fradkin, Field Theories of Condensed Matter Systems, Addison-Wesley, Reading, MA, 1991, and references therein.
[38] M. Peskin, Ann. Phys. (N.Y.) 113 (1978).
[39] T. Banks, R. Myerson, J.B. Kogut, Nucl. Phys. B 129 (1977) 493.
[40] S. Coleman, Phys. Rev. D 11 (1975) 2088.
[41] D.J. Amit, Y.Y. Goldschmidt, G. Grinstein, J. Phys. A 13 (1980) 585.
[42] J.M. Kosterlitz, D.J. Thouless, J. Phys. C 6 (1973) 1181;
J.M. Kosterlitz, J. Phys. C 7 (1974) 1046.
[43] J.V. José, L.P. Kadanoff, S. Kirkpatrick, D.R. Nelson, Phys. Rev. B 16 (1977) 1217.
[44] J.M. Kosterlitz, J. Phys. C 10 (1977) 3753;
See also: D.R. Nelson, Phys. Rev. B 26 (1982) 269.
[45] W. Chen, M.P.A. Fisher, Y. Wu, Phys. Rev. B 48 (1993) 13749.
[46] D.J. Amit, S. Elitzur, E. Rabinovici, R. Savit, Nucl. Phys. B 210 (1982) 69.
[47] D.R. Nelson, J.M. Kosterlitz, Phys. Rev. Lett. 39 (1977) 1201.
[48] J. Bartholomew, Phys. Rev. B 28 (1983) 5378.
[49] S. Mo, J. Hove, A. Sudbø, Phys. Rev. B 65 (2002) 104501;
See also: J. Hove, S. Mo, A. Sudbø, Phys. Rev. B 66 (2002) 064524.
[50] H. Kleinert, F.S. Nogueira, Phys. Rev. B 66 (2002) 012504.
[51] B.I. Halperin, T.C. Lubensky, S.-K. Ma, Phys. Rev. Lett. 32 (1974) 292;
J.-H. Chen, T.C. Lubensky, D.R. Nelson, Phys. Rev. B 17 (1978) 4274;
I.D. Lawrie, Nucl. Phys. B 200 (1982) 1.
[52] M. Kiometzis, H. Kleinert, A.M.J. Schakel, Phys. Rev. Lett. 73 (1994) 1975;
M. Kiometzis, H. Kleinert, A.M.J. Schakel, Fortschr. Phys. 43 (1995) 697.
[53] J.B. Marston, Phys. Rev. Lett. 64 (1990) 1166.

[54] R. Pisarski, Phys. Rev. D 29 (1984) 2423;
T.W. Appelquist, M. Bowick, D. Karabali, L.C.R. Wijewardhana, Phys. Rev. D 33 (1986) 3704.
[55] T. Appelquist, D. Nash, L.C.R. Wijewardhana, Phys. Rev. Lett. 60 (1988) 2575.
[56] E. Dagotto, J.B. Kogut, A. Kocić, Phys. Rev. Lett. 62 (1989) 1083.
[57] T. Appelquist, J. Terning, L.C.R. Wijewardhana, Phys. Rev. Lett. 75 (1995) 2081.
[58] T. Appelquist, A.G. Cohen, M. Schmaltz, Phys. Rev. D 60 (1999) 045003.
[59] S.J. Hands, J.B. Kogut, C.G. Strouthos, Nucl. Phys. B 645 (2002) 321.
[60] M. Franz, Z. Tešanović, Phys. Rev. Lett. 87 (2001) 257003.
[61] I.F. Herbut, Phys. Rev. Lett. 88 (2002) 047006;
Z. Tešanović, O. Vafek, M. Franz, Phys. Rev. B 65 (2002) 180511.
[62] D.J. Lee, I.F. Herbut, Phys. Rev. B 66 (2002) 094512;
O. Vafek, Z. Tešanović, M. Franz, Phys. Rev. Lett. 89 (2002) 157003.
[63] H.R. Fiebig, R.M. Woloshyn, Phys. Rev. D 42 (1990) 3520.
[64] X.-G. Wen, Phys. Rev. Lett. 88 (2002) 011602.
[65] A.P. Young, Phys. Rev. B 19 (1982) 1855.
[66] E.H. Hauge, P.C. Hemmer, Phys. Norvegica 5 (1971) 209.
[67] E. Fradkin, F.A. Schaposnik, Phys. Rev. Lett. 66 (1991) 276.
[68] M.C. Diamantini, P. Sodano, C.A. Trugenberger, Phys. Rev. Lett. 71 (1993) 1969;
M.C. Diamantini, P. Sodano, C.A. Trugenberger, Phys. Rev. Lett. 75 (1995) 3517.
[69] S.J. Rey, A. Zee, Nucl. Phys. B 352 (1991) 897.
[70] H. Kleinert, F.S. Nogueira, Duality and self-duality in Ginzburg–Landau theory with Chern–Simons term, cond-mat/0110513.
[71] B.I. Halperin, P.A. Lee, N. Read, Phys. Rev. B 47 (1993) 7312.