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Nematic world crystal model of gravity explaining absence of torsion in spacetime

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Abstract

We attribute the gravitational interaction between sources of curvature to the world being a crystal which has undergone a quantum phase transition to a nematic phase by a condensation of dislocations. The model explains why spacetime has no observable torsion and predicts the existence of curvature sources in the form of world sheets, albeit with different high-energy properties than those of string models.

1. Introduction

Present-day string models of elementary particles are based on the assumption that relativistic physics will prevail at all energy scales and, moreover, show recurrent particle spectra at arbitrary multiples of the Planck mass. Disappointed by the failure of these models [1] to explain correctly even the low-lying excitations, and the apparent impossibility of ever observing the characteristic recurrences, an increasing number of theoreticians is beginning to suspect that God may have chosen a completely different extension of present-day Lorentz-invariant physics to extremely high energies [2–4]. This philosophy has been advocated by one of the authors (H.K.) for almost two decades. In 1987, he proposed a simple three-dimensional Euclidean world crystal model of gravitation in which dislocations and disclinations represent curvature and torsion in the geometry of spacetime [5]. A full theory of gravity with torsion based on this picture is published in the textbook [6] (see also [7,8]).

The simple 1987 model had the somewhat unaesthetic feature that the crystal possessed only second-gradient elasticity to deliver the correct forces between the sources of curvature, which for an ordinary first-gradient elasticity grow linearly with the distance R and are thus confining. In this Letter we would like to point out that the correct $1/R$-behavior can also be obtained in an ordinary world crystal with first-gradient elasticity by assuming that the dislocations have proliferated. This explains also why the theory of general relativity requires only curvature for a correct description of gravitational forces, but no torsion. Such a state of the world crystal bears a close relationship with the

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nematic quantum liquid crystals of condensed matter physics, first suggested by Kivelson et al. [9], and believed to be of relevance both for the quantum Hall effect [10] and in high-T_c superconductors [11].

Our model will be formulated as before in three euclidean dimensions, for simplicity. The generalization to four dimensions is straightforward. The elastic energy is expressed in terms of a material *displacement field* $u_i(\mathbf{x})$ as

$$E = \int d^3x \left[\mu u_{ij}^2(\mathbf{x}) + \frac{\lambda}{2} u_{ii}^2(\mathbf{x}) \right], \tag{1.1}$$

where

$$u_{ij}(\mathbf{x}) \equiv \frac{1}{2} \left[\partial_i u_j(\mathbf{x}) + \partial_j u_j(\mathbf{x}) \right] \tag{1.2}$$

is the *strain tensor* and μ, ν are the elastic shear moduli. The elastic energy goes to zero for infinite wave length since in this limit $u_i(\mathbf{x})$ reduces to a pure translation under which the energy of the system is invariant. The crystallization process causes a spontaneous breakdown of the translational symmetry of the system. The elastic distortions describe the Nambu–Goldstone modes resulting from this symmetry breakdown. Note that so far the crystal has an extra longitudinal sound wave with a different velocity than the shear waves.

A crystalline material always contains defects. In their presence, the elastic energy depends only on the difference of the total distortion from the so-called plastic distortion $u_i^{\mathrm{p}}(\mathbf{x})$. If $u_{ij}^{\mathrm{p}}(\mathbf{x})$ denotes the *plastic part* of the strain tensor, the energy reads

$$E = \int d^3x \left[\mu \left(u_{ij} - u_{ij}^{\mathrm{p}} \right)^2 + \frac{\lambda}{2} \left(u_{ii} - u_{ii}^{\mathrm{p}} \right)^2 \right]. \tag{1.3}$$

In general, the crystal may contain a grand-canonical ensemble of line-like defects with a dislocation density

$$\alpha_{il} = \epsilon_{ijk} \partial_j \partial_k u_l^{\mathrm{p}}(\mathbf{x}) = \delta_i(\mathbf{x}; L)(b_l + \epsilon_{lqr} \Omega_q x_r), \tag{1.4}$$

and a *disclination density*

$$\begin{aligned} \theta_{il} &= \epsilon_{ijk} \partial_j \epsilon_{pmn} \left[\partial_n u_m^{\mathrm{p}}(\mathbf{x}) - \partial_m u_n^{\mathrm{p}}(\mathbf{x}) \right] \\ &= \delta_i(\mathbf{x}; L) \Omega_l, \end{aligned} \tag{1.5}$$

where b_l and Ω_l are the so-called Burgers and Franck vectors of the defects and $\delta_i(\mathbf{x}; L) \equiv \int_L d\bar{x}_i \, \delta(x - \bar{x})$ are δ-functions on the lines L.

The densities satisfy the conservation laws

$$\partial_i \alpha_{ik} = -\epsilon_{kmn} \theta_{mn}, \qquad \partial_i \theta_{il} = 0. \tag{1.6}$$

Dislocation lines are either closed or they end in disclination lines, and disclination lines are closed. These are Bianchi identities of the defect system.

An important geometric quantity characterizing dislocation and disclination lines is the *incompatibility* or *defect density*

$$\eta_{ij}(\mathbf{x}) = \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m u_{ln}^{\mathrm{p}}(\mathbf{x}). \tag{1.7}$$

It can be decomposed into disclination and dislocation density as follows [6]:

$$\begin{aligned} \eta_{ij}(\mathbf{x}) = \theta_{ij}(\mathbf{x}) + \frac{1}{2} \partial_m \big[& \epsilon_{min} \alpha_{jn}(\mathbf{x}) + (i \leftrightarrow j) \\ & - \epsilon_{ijn} \alpha_{mn}(\mathbf{x}) \big]. \end{aligned} \tag{1.8}$$

This tensor is symmetric and conserved

$$\partial_i \eta_{ij}(\mathbf{x}) = 0, \tag{1.9}$$

again a Bianchi identity of the defect system.

It is useful to separate from the dislocation density (1.4) the contribution from the disclinations which causes the nonzero right-hand side of (1.6). Thus we define a *pure dislocation density*

$$\alpha_{ij}^b(\mathbf{x}) \equiv \alpha_{ij}(\mathbf{x}) - \alpha_{ij}^{\Omega}(\mathbf{x}), \tag{1.10}$$

which satisfies $\partial_i \alpha_{ij}^b = 0$. Accordingly, we split

$$\eta_{ij}(\mathbf{x}) = \eta_{ij}^b(\mathbf{x}) + \eta_{ij}^{\Omega}(\mathbf{x}), \tag{1.11}$$

where

$$\eta_{ij}^b(\mathbf{x}) = \frac{1}{2} \left[\epsilon_{min} \alpha_{jn}^b(\mathbf{x}) + (i \leftrightarrow j) - \epsilon_{ijn} \alpha_{mn}^b(\mathbf{x}) \right], \tag{1.12}$$

and the pure disclination part of the defect tensor looks like (1.8), but with superscripts Ω on η_{ij} and α_{ij}.

The tensors α_{ij}, θ_{ij}, and η_{ij} are linearized versions of important geometric tensors in the *Riemann–Cartan space* of defects, a non-Euclidean space with curvature and torsion. Such a space can be generated from a flat space by a plastic distortion, which is mathematically represented by a *nonholonomic* mapping [7,8] $x_i \to x_i + u_i(\mathbf{x})$. Such a mapping is nonintegrable. The displacement fields and their first derivatives fail to satisfy the Schwarz integrability criterion:

$$\begin{aligned} &(\partial_i \partial_j - \partial_j \partial_i) u(\mathbf{x}) \neq 0, \\ &(\partial_i \partial_j - \partial_j \partial_i) \partial_k u_l(\mathbf{x}) \neq 0. \end{aligned} \tag{1.13}$$

The metric and the affine connection of the geometry in the plastically distorted space are $g_{ij} = \delta_{ij} + \partial_i u_j + \partial_j u_i$ and $\Gamma_{ijl} = \partial_i \partial_j u_l$, respectively. The noncommutativity of the derivatives in front of $u_l(\mathbf{x})$ implies a nonzero torsion, the torsion tensor being $S_{ijk} \equiv (\Gamma_{ijk} - \Gamma_{jik})/2$. The dislocation density α_{ij} is equal to $\alpha_{ij} = \epsilon_{ikl} S_{klj}$.

The noncommutativity of the derivatives in front of $\partial_k u_l(\mathbf{x})$ implies a nonzero curvature. The disclination density θ_{ij} is the Einstein tensor $\theta_{ij} = R_{ji} - \frac{1}{2} g_{ji} R$ of this Einstein–Cartan defect geometry. The tensor η_{ij}, finally, is the Belinfante symmetric energy–momentum tensor, which is defined in terms of the canonical energy–momentum tensor and the spin density by a relation just like (1.8). For more details on the geometric aspects see Part IV in Vol. II of [6], where the full one-to-one correspondence between defect systems and Riemann–Cartan geometry is developed as well as a gravitational theory based on this analogy.

Let us now show how linearized gravity emerges from the energy (1.3). For this we eliminate the jumping surfaces in the defect gauge fields from the partition function by introducing conjugate variables and associated stress gauge fields. This is done by rewriting the elastic action of defect lines as

$$E = \int d^3x \left[\frac{1}{4\mu} \left(\sigma_{ij}^2 - \frac{\nu}{1+\nu} \sigma_{ii}^2 \right) + i\sigma_{ij} \left(u_{ij} - u_{ij}^{\mathrm{p}} \right) \right], \tag{1.14}$$

where $\nu \equiv \lambda/2(\lambda + \mu)$ is Poisson's ratio, and forming the partition function by integrating the Boltzmann factor $e^{-E/k_B T}$ over σ_{ij}, u_i, and summing over all jumping surfaces S in the plastic fields. The integrals over u_i yield the conservation law $\partial_i \sigma_{ij} = 0$. This can be enforced as a Bianchi identity by introducing a stress gauge field h_{ij} and writing $\sigma_{ij} = G_{ij} \equiv \epsilon_{ikl} \epsilon_{jmn} \partial_k \partial_m h_{ln}$. The double curl on the right-hand side is recognized as the Einstein tensor in the geometric description of stresses, expressed in terms of a small deviation $h_{ij} \equiv g_{ij} - \delta_{ij}$ of the metric from the flat-space form. Inserting G_{ij} into (1.14) and using (1.7), we can replace the energy in the partition function by $E = E^{\mathrm{stress}} + E^{\mathrm{def}}$ where

$$E^{\mathrm{stress}} + E^{\mathrm{def}} \equiv \int d^3x \left[\frac{1}{4\mu} \left(G_{ij}^2 - \frac{\nu}{1+\nu} G_{ii}^2 \right) + i h_{ij} \eta_{ij} \right], \tag{1.15}$$

where the defect tensor (1.8) has the decomposition

$$\eta_{ij} = \eta_{ij}^{\Omega} + \partial_m \epsilon_{min} \alpha_{jn}^b. \tag{1.16}$$

The defects have also core energies which has been ignored so far in this continuum formulation of defects. They can properly been taken into account only in a lattice formulations. It has been shown in the textbook [6] that these give rise to leading quadratic terms in dislocation and disclination densities. If we focus attention on the dislocation part of the defect density (1.16), which is relevant for the phase transition to be studied, their fluctuations are governed by an energy

$$E^{\mathrm{disl}} = i \int d^3x \left(\epsilon_{imn} \partial_m h_{ij} \alpha_{jn}^b + \frac{\epsilon_c}{2} \left(\alpha_{jn}^b \right)^2 \right). \tag{1.17}$$

We now assume that the world crystal has undergone a transition to a phase in which dislocations are condensed. Actually, to reach such a state, whose existence was conjectured for two-dimensional crystals in Ref. [12], without a simultaneous condensation of disclinations in a first-order melting transition, the model requires a modification by an additional higher-gradient rotational energy. This was shown in [13] and verified by Monte Carlo simulations in [14]. The three-dimensional extension of the extended model is described in [6]. The expressions are to long to be written down in the Letter, and their explicit forms are not relevant for the present discussion. Here we only need the outcome of such a construction that it is possible to have an elastic plus plastic energy which allows for a phase transition in which only dislocations condense while the disclinations remain dilute.

The condensed phase is described by a partition function in which the discrete sum over the pure dislocation densities in α_{jn}^b is approximated by an ordinary functional integral. This has been shown in Ref. [7]. The general integration rule is

$$\int d^3l \, \delta(\boldsymbol{\partial} \cdot \mathbf{l}) \exp\left(\frac{-\beta \mathbf{l}^2}{2} + i \mathbf{l}\mathbf{a} \right) = \exp\left(\frac{-\mathbf{a}_T^2}{2\beta} \right), \tag{1.18}$$

where $\mathbf{a}_T$ has the components

$$a_{Ti} \equiv -i \epsilon_{ijk} \partial_j a_k / \sqrt{-\boldsymbol{\partial}^2}.$$

The Boltzmann factor resulting in this way from E^{stress} plus (1.17) has now the energy

$$E' = \int d^3x \left[\frac{1}{4\mu} \left(G_{ij}^2 - \frac{\nu}{1+\nu} G_{ii}^2 \right) + \frac{1}{2\epsilon_c} G_{ij} \frac{1}{-\boldsymbol{\partial}^2} G_{ij} \right]. \tag{1.19}$$

The second term implies a Meissner-like screening of the initially confining gravitational forces between the disclination part of the defect tensor to Newton-like forces. For distances longer than the Planck scale, we may ignore the stress term and find the effective gravitational action for the disclination part of the defect tensor:

$$E \approx \int d^3x \left(\frac{1}{2\epsilon_c} G_{ij} \frac{1}{-\boldsymbol{\partial}^2} G_{ij} + i h_{ij} \eta_{ij}^{\Omega} \right). \tag{1.20}$$

A path integral over h_{ij} and a sum over all line ensembles applied to the Boltzmann factor $e^{-E/\hbar}$ is a simple Euclidean model of pure quantum gravity. The line fluctuations of η_{ij}^{Ω} describe a fluctuating Riemann geometry perforated by a grand-canonical ensemble arbitrarily shaped lines of curvature. As long as the loops are small they merely renormalize the first term in the energy (1.20). Such effects were calculates in closely related theories in great detail in Ref. [21]. They also give rise to post-Newtonian terms in the above linearized description of the Riemann space.

We may now add matter to this gravitational environment. It is coupled to h_{ij} by the usual Einstein interaction

$$E^{\text{int}} \approx \int d^3x \, h_{ij} T^{ij}, \tag{1.21}$$

where T^{ij} is the symmetric Belinfante energy–momentum tensor of matter. Inserting for G_{ij} the double-curl of h_{ij} we see that the energy (1.20) produces the correct Newton law if the core energy is $\epsilon_c = 8\pi G$, where G is Newton's constant.

Note that the condensation process of dislocations has led to a pure Riemann space without torsion. Just as a molten crystal shows residues of the original crystal structure only at molecular distances, remnants of the initial torsion could be observed only near the Planck scale. This explains why present-day general relativity requires only a Riemann space, not a Riemann–Cartan space.

In the non-relativistic context, a dislocation condensate is characteristic for a nematic liquid crystal, whose order is translationally invariant, but breaks rotational symmetry (see [6,12] in the two dimensions and [15] in the $(2+1)$-dimensional quantum theory). The Burgers vector of a dislocation is a vectorial topological charge, and nematic order may be viewed as an ordering of the Burgers vectors in the dislocation condensate. Such a manifest nematic order would break the low energy Lorentz-invariance of spacetime. We may, however, imagine that the stiffness of the directional field of Burgers vectors is so low that, by the criterion of Ref. [16], they have undergone a Heisenberg-type of phase transition into a directionally disordered phase in an environment with only a few disclinations. In three dimensions, dislocations (and disclinations) are line-like. This has the pleasant consequence, that they can be described by the disorder field theories developed in [17] in which the proliferation of disclinations follows the typical Ginzburg–Landau pattern of the field expectation acquiring a nonzero expectation value. A cubic interaction becomes isotropic in the continuum limit [18] (this is the famous fluctuation-induced symmetry restoration of the Heisenberg fixed point in a ϕ^4-theory with O(3)-symmetric plus cubic interactions [19]). The isotropic phase is similar to the *topological* form of nematic order identified by Lammert et al. [20] as the Coulomb phase in the generalized Z_2 gauge theory of nematic order: rotational (Lorentz) invariance is restored even though there is no condensate of disclinations. A similar isotropic phase is also found in the theory of non-relativistic elasticity in $2+1$ dimensions [15] where one learns that any microscopic rotational anisotropy will render the topological order unstable towards a full nematic order. The world crystal should certainly lose all information on its crystal axes.

The generalization to four Euclidean spacetime dimensions changes mainly the geometry of the defects. In four dimensions, they become world sheets, and a second-quantized disorder field description of surfaces has not yet been found. But the approximation of representing a sum over dislocation surfaces in the proliferated phase as an integral as in Eq. (1.18) will remain valid, so that the above line of arguments will survive, this being a natural generalization of the Meissner–Higgs mechanism. The disclination sources of curvature will be world sheets, as an attractive feature for string theorists. However, the high-energy properties will be completely different. On the one hand, these surfaces behave nonrelativistically as the energies approach the Planck scale, on the other hand, they will not have the characteristic multi-Planck recurrences of the common strings. Although the latter property may never be verified in the laboratory, the deviations from relativity at high energies or short distances may come into experimentalists reach in the possibly distant future.

Note that our model has automatically a vanishing cosmological constant. Since the atoms in the crystal are in equilibrium, the pressure is zero. This explanation is similar to that given by Volovik [2] with his helium droplet analogies.

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