# Quantum electrodynamics in $2+1$ dimensions, confinement, and the stability of $U(1)$ spin liquids


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Compact quantum electrodynamics in $2+1$ dimensions often arises as an effective theory for a Mott insulator, with the Dirac fermions representing the low-energy spinons. An important and controversial issue in this context is whether a deconfinement transition takes place. We perform a renormalization group analysis to show that deconfinement occurs when $N > N_c = 36/\pi^3 \approx 1.161$, where $N$ is the number of fermion replica. For $N < N_c$, however, there are two stable fixed points separated by a line containing a unstable non-trivial fixed point: a fixed point corresponding to the scaling limit of the non-compact theory, and another one governing the scaling behavior of the compact theory. The string tension associated to the confining interspinon potential is shown to exhibit a universal jump as $N \to N_c^-$. Our results imply the stability of a spin liquid at the physical value $N = 2$ for Mott insulators.



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An important topic currently under discussion in condensed matter physics community is the emergence of deconfined quantum critical points in gauge theories of Mott insulators in $2+1$ dimensions [1, 2]. A closely related problem concerns the stability of $U(1)$ spin liquids in $2+1$ dimensions [3, 4]. In either case, models which are often considered as toy models in the high-energy physics literature are supposed to describe the low-energy properties of real systems in condensed matter physics. For instance, a model that frequently appears in the condensed matter literature is the $(2+1)$-dimensional quantum electrodynamics (QED3) [5, 6]. It emerges, for instance, as an effective theory for Mott insulators [7–9]. Let us briefly recall how it arises in this context. The Hamiltonian of a $SU(N)$ Heisenberg antiferromagnet is written in a slave-fermion representation as $H = -(J/N)\sum_{\langle i,j\rangle} f_{i\alpha}^\dagger f_{j\alpha} f_{j\beta}^\dagger f_{i\beta}$, where the local constraint $f_{i\alpha}^\dagger f_{i\alpha} = N/2$ holds. A Hubbard-Stratonovich transformation introduces the auxiliary field $\chi_{ij} = \langle f_{i\alpha}^\dagger f_{j\alpha}\rangle$ [7]. The resulting effective theory can be treated as a *lattice gauge theory*, where the gauge field $A_{ij}$ emerges as the phase of $\chi_{ij}$, i.e., $\chi_{ij} = \chi_0 e^{iA_{ij}}$, where $\chi_0$ is determined from mean-field theory. The $(2+1)$-dimensional low-energy effective Lagrangian in imaginary time has the form [4, 7–9]

$$\mathcal{L} = \frac{1}{4e_0^2} F_{\mu\nu}^2 + \sum_{a=1}^{N} \bar{\psi}_a \gamma_\mu (\partial_\mu + iA_\mu)\psi_a, \tag{1}$$

where each $\psi_a$ is a four-component Dirac spinor and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual field strength tensor. A rough estimate of the bare gauge coupling is given by $e_0^2 \sim \chi_0^4 a^3$, where $a$ is the lattice spacing.

An anisotropic version of QED3 has also been studied in the context of phase fluctuations in $d$-wave superconductors [10, 11]. A key feature of the QED3 theory of Mott insulators is its parity conservation. In fact, it is possible to introduce two different QED3s, one which conserves parity and one which does not. The latter theory involves two-component spinors, and allows for a *chirally-invariant* mass term which is not parity-invariant. In such a QED3 theory a Chern-Simons term [12] is generated by fluctuations [13]. The QED3 theory relevant to Mott insulators and $d$-wave superconductors involves four-component spinors, and does possess chiral symmetry [5, 6]. In such a model, the chiral symmetry can be spontaneously broken through the dynamical generation of a fermion mass. In the context of Mott insulators, the chiral symmetry breaking corresponds to the development of Néel order [9]. Indeed, the nonzero condensate $\langle\bar{\psi}\psi\rangle$ corresponds to the staggered magnetization. Since in condensed matter physics parity-conserving QED3 is not just a toy model, and that the four-component Dirac spinors represent physical excitations — the low-energy spinons, we may call this theory *quantum spinodynamics.*

An important feature of QED3 for Mott insulators is that the $U(1)$ gauge group is *compact.* The compactness causes important changes in the physical properties of the theory. It allows for quantum excitation of magnetic monopoles which play an important role in determining the phase structure of the theory. This has been known for a long time. In particular, Polyakov [14] has shown that compact Maxwell theory in $2+1$ dimensions confines permanently electric test charges. The electrostatic potential has the form $V(R) \sim R$, instead of the usual two-dimensional Coulomb potential $V(R) \sim \ln R$ of the non-compact Maxwell theory in $2+1$ dimensions. Since $V(R) \sim R$ holds for all values of the gauge coupling, the compact $(2+1)$-dimensional Maxwell theory does not exhibit any phase transition, i.e., the confinement is permanent. This theory is equivalent to a Coulomb gas of magnetic monopoles in three dimensions and it is well known that such a gas does not undergo any phase transition. However, when matter fields are included the situation changes, and a deconfinement transition may occur. Indeed, matter fields induce shape fluctuations of the *electric* flux tube, leading to a correction term to the linearly confining potential. Using a string model for the electric flux tube, Lüscher [15] found

$$V(R) = \sigma R - \frac{(d-2)\pi}{24R} + \mathcal{O}(1/R^2), \tag{2}$$

where $\sigma$ is the string tension. At the deconfining critical point, the string tension vanishes and only the Lüscher term remains at long distances. It represents the "black-body" energy of the $(d-2)$ transverse fluctuations of the two-dimensional worldsheet of the string [16].

Interestingly, by studying QCD near four dimensions, Peskin [17] found that, at the critical point, the interquark potential does have the $1/R$-behavior for all $d \in (4, 4+\epsilon)$, and argued that this should also be valid outside this small dimension interval. Recently we [18] have found that such a behavior is also realized in an Abelian U(1)-gauge theory for $d \in (2,4)$, provided that this is coupled to matter fields.

In order to better illustrate this mechanism, we shall explicitly perform the calculation of the interspinon potential to one-loop order in arbitrary space-time dimension $2 < d \leq 4$, going to the case of interest $d = 3$ at the end [19]. The potential is defined by

$$V(R) = -e_0^2 \int \frac{d^{d-1}q}{(2\pi)^{d-1}} \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{\mathbf{q}^2[1+\Pi(\mathbf{q})]}, \tag{3}$$

where $e_0$ is the bare electric charge, and $\Pi(q)$ is the vacuum polarization. At one-loop order, the vaccum polarization is given by $\Pi(q) = 8A(d)Ne_0^2|q|^{d-4}$, with $A(d) = \Gamma(2-d/2)\Gamma^2(d/2)/[(4\pi)^{d/2}\Gamma(d)]$. At large distances the vacuum polarization gives the more relevant contribution to the scaling behavior if $2 < d < 4$, and the interspinon potential is given by

$$V(R) = -\frac{1}{2^{d+1}\pi^{(d-2)/2}\Gamma(d/2-1)A(d)N}\frac{1}{R}. \tag{4}$$

For $d = 3$ the above potential becomes simply $V(R) = -4/(\pi N R)$. Interestingly, by expanding (4) near $d = 2$, we obtain at lowest order

$$V(R) \approx -\frac{(d-2)\pi}{8NR}, \tag{5}$$

which has for $N = 3$ replica precisely the form of the Lüscher term, although this theory has no confinement. In order to allow for this, we compatify the U(1) gauge group which gives rise to magnetic monopoles. In the absence of fermions this theory in $d = 3$ is known to be described via a duality transformation by the sine-Gordon Lagrangian [14]

$$\mathcal{L} = \frac{1}{2}\left(\frac{e_0}{2\pi}\right)^2 (\partial_\mu \chi)^2 - 2z_0 \cos\chi. \tag{6}$$

This is also the field theory of a Coulomb gas of monopoles, with $z_0$ being the bare fugacity of the gas. The RG equations for a Coulomb gas in $d = 3$ were obtained a long time ago by Kosterlitz [20]. His results can be used here to obtain the RG equation for the gauge coupling in the absence of fermionic matter. By introducing the dimensionless couplings $f \equiv e^2(l)/e_0^2$ and $y \equiv z(l)/(e_0^2)^3$, where $l = \ln(e_0^2 r)$ is a logarithmic length scale, we obtain

$$\frac{df}{dl} = 4\pi^2 y^2 + f, \tag{7}$$

$$\frac{dy}{dl} = \left(3 - \frac{\pi^3}{f}\right) y. \tag{8}$$

The above equations imply that there is no fixed point for the gauge coupling. Therefore, the compact three-dimensional Maxwell theory does not undergo any phase transition. The photon mass $M^2 = 8\pi^2 z/e^2$ is always non-zero and the theory confines permanently the electric charges. It can be seen from Eq. (7) that a kind of *anti-screening* happens in this theory, which is responsible for confinement. Indeed, we can rewrite Eq. (7) as $df/dl = (1-\hat{\gamma}_A)f$, with $\hat{\gamma}_A = -4\pi^2 y^2/f$. The negative sign of $\hat{\gamma}_A$ is actually a remarkable example of the intimate link between asymptotic freedom and confinement [21].

Next we obtain the modification to Eqs. (7) and (8) due to the coupling with the matter fields. In order to derive the RG equations including matter we have employed a formalism similar to the one developed by Young [22] in the case of the two-dimensional Coulomb gas. This formalism is based on a mean-field self-consistent approximation and applies very well to the $d > 2$ case, since $d = 2$ is the upper critical dimension for the Coulomb gas. The needed modification comes from the extra renormalization of the gauge coupling due to the vacuum polarization. This leads to an effective charge $e^2(l) = \varepsilon(e^l/e_0^2)Z_A(l)e^l e_0^2$, where $\varepsilon(r)$ is the scale-dependent "dielectric" constant of the Coulomb gas of magnetic monopoles, and $Z_A(l)$ is the gauge field wave function renormalization. In terms of dimensionless couplings this leads to

$$\frac{df}{dl} = 4\pi^2 y^2 + (1-\gamma_A)f, \tag{9}$$

where $\gamma_A \equiv -d\ln Z_A/dl$. However, the expression of $y^2$ in terms of bare variables is different from before, being given by $y^2 = (32\pi^2/3)z_0^2 Z_A(l)e^{6l-u(l)}/(e_0^2)^6$, where $u(l) \equiv U(e^l/e_0^2)$ is a self-consistent magnetic monopole potential satisfying $du/dl = \pi^3/f$ [23]. Therefore, the coupling to matter modify also Eq. (8) to

$$\frac{dy}{dl} = \left(3 - \frac{\pi^3}{f} - \frac{\gamma_A}{2}\right) y. \tag{10}$$

Note the crucial difference between the analysis made here and the one of Refs. [18], [24] and [25]. *There it was* *assumed that the underlying non-compact theory is critical, and monopoles were introduced only at that point. This corresponds to take the RG function $\gamma_A$ at the fixed point of the non-compact theory, i.e., $\gamma_A^* \equiv \eta_A = 1$ [26] for $d = 3$.* In this way, Eqs. (9) and (10) become similar to the RG equations of a Kosterlitz-Thouless phase transition [27], except that the present dimensionality is three instead of two [24, 28]. Our Eqs. (9) and (10) have the advantage of being valid at all length scales. Eqs. (9) and (10) are similar to the ones in the work of Hermele *et al.* [4]. There is, however, an important difference: Eq. (10) contains the correction proportional to $\gamma_A$ which is absent in Ref. [4]. This will allow us to strenghten considerably the results obtained by these authors. It is important to emphasize that the additional term in Eq. (10) *cannot* be neglected even if a large $N$ limit is assumed. Indeed, since the large $N$ limit is taken for $Ne_0^2$ fixed, it follows that $\gamma_A \sim \mathcal{O}(1)$, as it should be, since it gives the anomalous dimension RG function of the non-compact theory. It is of the same order as the first term between parentheses in Eq. (10), which corresponds to the dimensionality of the space-time. Thus, this problem has no obvious control parameter, and should be seen as a matter field fluctuation-corrected Debye-Hückel theory. In the case of compact Maxwell theory, the Debye-Hückel theory corresponds to a non-dilute gas of monopoles and its validity is determined the parameter $n\lambda_D^3$, where $n$ is the monopole density and $\lambda_D \equiv \sqrt{e^2/(4\pi^2 n)}$ is the Debye Length. For the compact Maxwell theory we have that $n\lambda_D^3 \gg 1$ and the Debye-Hückel theory is a very good approximation. Including matter fields makes the monopole gas dilute and $n\lambda_D^3$ is no longer large. The Debye-Hückel parameter can be written as $n\lambda_D^3 = \sqrt{2}e^2/(8\pi^2 M)$, where $M = 2\pi\sqrt{2z}/e$ is the photon mass. Thus, in the presence of matter a perturbation theory around the compact Maxwell theory can be performed where $e^2/M$ is a small parameter. We will see below that the fixed points at nonzero fugacity give indeed a small value of $n\lambda_D^3$.

By considering the one-loop result $\gamma_A = Nf/8$, we find besides the fixed points $f_* = 8/N$ and $y_* = 0$ of the non-compact theory, the following non-trivial fixed points governing the phase structure of compact QED3:

$$f_\pm = \frac{4}{N}\left(6 \pm \sqrt{36 - N\pi^3}\right), \tag{11}$$

$$y_\pm = \frac{1}{\pi}\left(\frac{60 - N\pi^3 \pm 10\sqrt{36 - N\pi^3}}{2N}\right)^{1/2}. \tag{12}$$

The above fixed points exist only for $N < N_c = 36/\pi^3 \approx 1.161$. For $N > N_c$ only the non-compact fixed point exists. For $N = 1$ the fixed points $(f_\pm, y_\pm)$ give for the Debye-Hückel parameter the values $(n\lambda_D^3)_+ \approx 0.3$ and $(n\lambda_D^3)_- \approx 0.15$, respectively. In Fig. 1 we show a schematic flow diagram for the case $N = 1$. The dashed line in the flow diagram passes through the *unstable* fixed point having coordinates $(f_-, y_-)$. This line separates two different critical regimes. Note that $\gamma_A$ is $N$-dependent at the fixed points $(f_\pm, y_\pm)$, with its two possible values given by $\gamma_A^\pm = Nf_\pm/8 = (6 \pm \sqrt{36 - N\pi^3})/2$. *This result implies that there is no $N \geq 1$ for which $\gamma_A^\pm = 1$ in compact QED3.* This rules out a KT-like transition in compact QED3 for physical values of $N$.

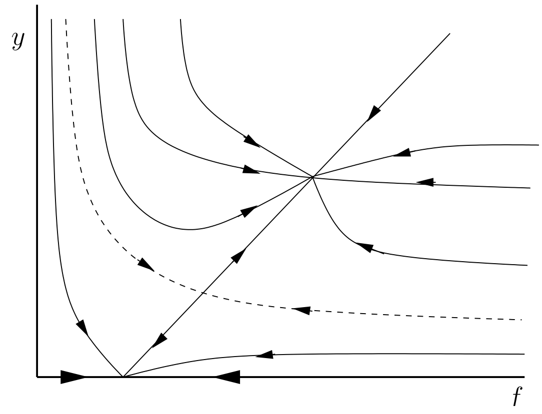


FIG. 1: Schematic flow diagram for the case $N = 1$

The flow diagram in Fig. 1 indicates two distinct physical regimes governed by stable fixed points separated by the dashed line in the figure. Depending on the initial conditions on the physical parameters, the system will choose to flow either to the non-compact fixed point below the dashed line, or to the compact one above the dashed line. The interesting physical regime for us is governed by the fixed points at nonzero fugacity. It is clear that the fixed points $(f_\pm, y_\pm)$ are associated with confined phases, since there both the photon mass $M$ and string tension $\sigma = 2e^2M/\pi^2$ are nonzero. The string tension approaches a universal value as $N$ approaches $N_c$ from the left, i.e., $\lim_{N\to N_c^-} \sigma/e_0^4 = 8(\pi/3)^{3/4}$. Since for $N > N_c$ the string tension vanishes, it follows that there is a universal jump at $N_c$. Thus, in the present context the string stiffness behaves similarly to the superfluid stiffness in two-dimensional superfluids [29], though here there is no KT transition. The vanishing of the string tension above $N_c$ is a clear signature for spinon deconfinement for $N = 2$.

Below $N_c$ the interspinon potential has the form $V(R) = \sigma R - \alpha/R + \mathcal{O}(1/R^2)$, where $\alpha$ is the universal coefficient of the Lüscher term for the string fluctuation in compact QED3. The coefficient $\alpha$ is defined by $\alpha = f_c/2\pi$, where $f_c$ is any of the three charged fixed points in Fig. 1. For the stable confining regime governed by the fixed point $(f_+, y_+)$ we obtain that $\alpha = 2(6 + \sqrt{36 - N\pi^3})/\pi N$.

It is perfectly plausible to argue that in the confined phase of compact QED3 the chiral symmetry is broken, just as in the QCD case [30]. Chiral symmetry breaking is believed to occur in QED3 for $N < N_{\rm ch}$, where typically $N_{\rm ch} \sim 3$. Indeed, an early estimate based on the analysis of the Schwinger-Dyson equation gives $N_{\rm ch} = 32/\pi^2 \approx 3.2$ [6], which was roughly confirmed by a Monte Carlo simulation giving $N_{\rm ch} = 3.5 \pm 0.5$ [31]. However, the true value of $N_{\rm ch}$ is still far from being consensual. For instance, recent Monte Carlo simulations do not find a decisive indication that chiral symmetry is broken for $N \geq 2$ [33] and an elaborate analysis of the Schwinger-Dyson equations gives $N_{\rm ch} \approx 4$ [34]. Our results indicate that for $N_c < N < N_{\rm ch}$ the spinons are deconfined but chiral symmetry is broken. However, it is not excluded that $N_c = N_{\rm ch}$. Recently, a conjectured inequality was used to suggest that $N_{\rm ch} = 3/2$ in the non-compact case [32]. It is remarkable that our critical value $N_c$ is so close to the latter estimate. If $N_{\rm ch} > 2$, we would obtain that for the physical case $N = 2$ antiferromagnetism is present [9], while the spinons are deconfined. In such a situation doping will eventually destroy the magnetic order and, since the spinons are deconfined, a genuine spin liquid will develop. Our results confirm the analysis of Ref. [4], whose discussion was made in the large $N$ limit.

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