# Equivalence Principle and Field Quantization in Curved Spacetime

H. Kleinert

*Institut für Theoretische Physik, Freie Universität Berlin,*
*Arnimallee 14, D14195 Berlin, Germany*
*and*
*ICRANeT, Piazzale della Republica 1,*
*10 -65122, Pescara, Italy*

To comply with the equivalence principle, fields in curved spacetime can be quantized only in the neighborhood of each point, where one can construct a freely falling *Minkowski* frame with *zero* curvature. In each such frame, the geometric forces of gravity can be replaced by a selfinteracting spin-2 field, as proposed by Feynman in 1962. At *any fixed* distance $R$ from a black hole, the vacuum in each freely falling volume element acts like a thermal bath of all particles with Unruh temperature $T_U = \hbar GM/2\pi cR^2$. At the horizon $R = 2GM/c^2$, the falling vacua show the Hawking temperature $T_H = \hbar c^3/8\pi GMk_B$.

1.) When including Dirac fermions into the theory of gravity, it is important to remember that the Dirac field is initially defined only by its transformation behavior under the Lorentz group. The invariance under general coordinate transformations can be incorporated only with the help of a vierbein field $e^\alpha{}_\mu(x)$ which couples Einstein indices with Lorentz indices $\alpha$. These serve to define anholonomic coordinate differentials $dx^\alpha$ in curved spacetime $x^\mu$:

$$dx^\alpha = e^\alpha{}_\mu(x)dx^\mu, \tag{1}$$

which at any given point have a Minkowski metric:

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta, \quad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\alpha\beta}. \tag{2}$$

With the help of the vierbein field one can write the action simply as [1]

$$\mathcal{A} = \int d^4x \sqrt{-g}\bar{\psi}(x)[\gamma^\alpha e_\alpha{}^\mu(x)(i\partial_\mu - \Gamma_\mu^{\alpha\beta}\tfrac{1}{2}\Sigma_{\alpha\beta}) - m]\psi(x), \quad g_{\mu\nu}(x) \equiv \eta_{\alpha\beta}e^\alpha{}_\mu(x)e^\beta{}_\nu(x), \tag{3}$$

where $\Sigma_{\alpha\beta}$ is the spin matrix, which is formed from the commutator of two Dirac matrices as $i[\gamma_\alpha,\gamma_\beta]/4$, and $\Gamma_\mu^{\alpha\beta} \equiv e^\alpha{}_\nu e^\beta{}_\lambda \Gamma_\mu{}^{\nu\lambda}$ is the spin connection, It is constructed from combination of the so-called objects of anholonomy $\Omega_{\mu\nu}{}^\lambda = \tfrac{1}{2}[e_\alpha{}^\lambda\partial_\mu e^\alpha{}_\nu - (\mu \leftrightarrow \nu)]$, by taking the sum $\Omega^{\mu\nu\lambda} - \Omega^{\nu\lambda\mu} + \Omega^{\lambda\mu\nu}$ and lowering two indices with the help of the metric $g_{\mu\nu}(x)$.

The theory of quantum fields in curvilinear spacetime has been set up on the basis of this Lagrangian, or a simpler version for bosons which we do not write down. The classical field equation is solved on the background metric $g_{\mu\nu}(x)$ in the entire spacetime. The field is expanded into the solutions, and the coefficients are quantized by canonical commutation rules, after which they serve as creation and annihilation operators on some global vacuum of the quantum system.

The purpose of this note is to make this this procedure compatible with the equivalence principle.

2.) If one wants to quantize the theory in accordance with the equivalence principle one must introduce creation and annihilation operators of proper elementary particles. These, however, are defined as irreducible representations of the Poincaré group with a given mass and spin. The symmetry transformations of the Poincaré group can be performed only in a Minkowski spacetime. According to Einstein's theory, and confirmed by Satellite experiment, we can remove gravitational forces locally at one point. The neighborhood will still be subject to gravitational fields. For the definition of elementary particles we need only a small neighborhood. In it, the geometric forces can be replaced by the forces coming from the spin-2 gauge field theory of gravitation, which was developed by R. Feynman in his 1962 lectures at Caltech [2]. This can be rederived by expanding of the metric in powers its deviations from the flat Minkowski metric. We define a Minkowski frame $x^a$ around the point of zero gravity, and extend it to an entire finite box without spacetime curvature. Inside this box, particle experiments can be performed and the transformation properties under the Poincaré group can be identified.

Inside the box, the fields are governed by the flat-spacetime action

$$\mathcal{A} = \int d^4x \sqrt{-g}\, \bar{\psi}(x)\{\gamma^a e_a{}^b(i\partial_b - \Gamma_a^{bc}\tfrac{1}{2}\Sigma_{bc}) - m\}\psi(x). \tag{4}$$

In this expression, $e_a{}^b = \delta_a{}^b + \tfrac{1}{2}h_a{}^b(x)$. The metric and the spin connection are defined as above, exchanging the indices $\alpha, \beta, \ldots$ by $a, b, \ldots$. All quantities must be expanded in powers of $h_a{}^b$.

Thus we have arrive at a standard local field theory in the freely falling Minkowski laboratory around the point of zero gravity. This action is perfectly Lorentz invariant, and the Dirac field can now be quantized without problems, producing an irreducible representation of the Poincaré group with states of definite momenta and spin orientation $|\mathbf{p}, s_3[m, s]\rangle$.

The Lagrangian governing the dynamics of the field $h_a{}^b(x)$ is well known from Feynman's lecture [2]. If the laboratory is sufficiently small, we may work with the Newton approximation:

$$\mathcal{A}^h = -\frac{1}{8\kappa}\int d^4x\, h_{ab}\epsilon^{cade}\epsilon_c{}^{bfg}\partial_d\partial_f h_{eg} + \ldots\, , \quad \kappa = 8\pi G/c^3, \;\; G = \text{Newton constant}, \tag{5}$$

where $\epsilon^{cade}$ is the antisymmetric unit tensor. If the laboratory is larger, for instance, if it contains the orbit of the planet mercury, we must include also the first post-Newtonian corrections.

Thus, although the Feynman spin-2 theory is certainly not a valid replacement of general relativity, it is so in a neighborhood of any freely falling point.

The vacuum of the Dirac field is, of course, not universal. Each point $x^\mu$ has its own vacuum state restricted to the associated freely falling Minkowski frame.

3.) There is an immediate consequence of this quantum theory. If we consider a Dirac field in a black hole, and go to the neighborhood of any point, the quantization has to be performed in the freely falling Minkowski frame with smooth forces. These are incapable of creating pairs. An observer at a fixed distance $R$ from the center, however, sees the vacua of these Minkowski frames pass by with acceleration $a = GM/R^2$, where $G$ is Newton's constant. At a given $R$, the frequency factor $e^{i\omega t}$ associated with the zero-point oscillations of each scalar particle wave of the world will be Doppler shifted to $e^{i\omega e^{at/c}/a}$, and this wave has frequencies distributed with a probabilty that behaves like $1/(e^{2\pi\Omega c/a} - 1)$. Indeed, if we Fourier-analyze this wave [3]:

$$\int_\infty^\infty dt\, e^{i\Omega t}e^{i\omega e^{at/c}/a} = e^{-\pi\Omega c/2a}\Gamma(i\Omega c/a)e^{-i\Omega c/a\log(\omega c/a)}(c/a). \tag{6}$$

we see that the probability to find the frequency $\Omega$ is $|e^{-\pi\Omega c/2a}\Gamma(i\Omega c/a)c/a|^2$, which is equal to $2\pi c/(\Omega a)$ times $1/(e^{2\pi\Omega c/a} - 1)$. The latter is a thermal Bose-Einstein distribution with an Unruh temperature $T_U = \hbar a/2\pi c k_B$ [4], where $k_B$ is the Boltzmann constant. The particles in this heat bath can be detected by suitable particle reactions as described in Ref. [5].

The Hawking temperature [6] $T_H$ is equal to the Unruh temperature of the freely falling Minkowski vacua at the surface of the black hole, which lies at the horizon $R = r_S \equiv 2GM/c^2$. There the Unruh temperature is equal to $T_U|_{a=GM/R^2, R=2GM/c^2} = \hbar c^3/8\pi GMk_B = T_H$.

Note that there exists a thermal bath of nonzero Unruh temperature $T_U(R) = T_H r_S^2/R^2$ at *any* distance $R$ from the center — even on the surface of the earth, where the temperature is $4\times 10^{-20}\,$K, too small to be measurable. In the light of this it is surprising that the derivation of the thermal bath from semiclassical pair creation is based on a coordinate singularity at the horizon [7].

For decreasing $R$ inside the horizon, the temperature rises to infinity, but this radiation cannot reach any outside oberver.

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[1] Our notation follows
H. Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation*, World Scientific, Singapore 2009, pp. 1–497 (http://www.physik.fu-berlin.de/~kleinert/b11).
The only exception is that the vierbein field is here called $e^\alpha{}_\mu$ rather than $h^\alpha{}_\mu$ to have the notation $h^a{}_b$ free for the small deviations of $e^a{}_b$ from the flat limit $\delta^a{}_b$.

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