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PREDICTIONS ON THE COUPLINGS OF f- AND ρ- TRAJECTORIES TO MESONS FROM CHIRAL SYMMETRY

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Abstract: We employ a chiral saturation scheme involving **15**, $L = 0$, and **15**, $L = 1$ mesons in a classification according to SU(4) $\otimes$ O(3) (i.e. the mesons π, ρ, ω, A_1, A_2, f, A_0 (δ), B, D and σ) and calculate all f and ρ Regge couplings between these mesons *via* a matrix version of finite-energy sum rules. In addition the size of exotic exchanges is evaluated and found to be small. The results are compared with experimental numbers and other models as far as available.

1. Introduction

Within a narrow resonance approximation to hadronic amplitudes, algebraic treatments of the chiral SU(2) $\otimes$ SU(2) charge algebra have provided us with a large number of pionic coupling constants [1,2]. For example, all couplings of pions to the mesons π, ρ, ω and A_1, A_2, f, A_0 (δ), B, D, σ, belonging to the **15**, $L = 0$, and **15**, $L = 1$ representations of SU(4) $\otimes$ O(3) can be found in the literature [3]. Similarly, large coupling schemes have been developed for most of the well-established baryon resonances, in particular, for the **56**, $L = 0$, **70**, $L = 1$, and **56**, $L = 2$ states of SU(6)$\otimes O_3$ (ref. [4]).

The calculation of such coupling schemes proceeds *via* the construction of reducible representations of the SU(2) $\otimes$ SU(2) commutators

$$[T_a, T_b] = i\epsilon_{abc} T_c \ , \tag{1.1}$$

$$[T_a, X_b(\lambda)] = i\epsilon_{abc} X_c(\lambda) \ , \tag{1.2}$$

$$[X_a(\lambda), X_b(\lambda)] = i\epsilon_{abc} T_c \ , \tag{1.3}$$

where, as usual, T_a denotes the isospin and $X_a(\lambda)$ the collinear matrix elements of the axial charges at infinite momentum and helicity λ [2]:

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$$[X_a(\lambda)]_{\beta\alpha}\, 2p_0 (2\pi)^3 \delta^3 (p_\beta - p_\alpha) = \lim_{p\to\infty} \langle p,\beta,\lambda | Q_a | p,\alpha,\lambda\rangle$$

The possible irreducible chiral contents are assigned to the particles in two steps:

(i) One assumes a non-relativistic quark model with SU(4) ⊗ O(3) symmetry. This group contains an SU(2) ⊗ SU(2) subgroup formed by the operators

$$\int d^3x\, q^\dagger \tfrac{1}{2}\tau_a q \ , \qquad (1.4)$$

$$\int d^3x\, q^\dagger \tfrac{1}{2}\tau_a \sigma_3 q \ . \qquad (1.5)$$

The matrix element of these operators between states at rest* are taken as a lowest order approximation to the algebra (1.1) – (1.3).

(ii) One accounts for the effects of symmetry breaking and of relativistic corrections by mixing the above representations and by identifying the mixed charges (1.5) with the matrix elements of $X_a(\lambda)$ in the infinite momentum frame.

As a consequence of eq. (1.3), the coupling constants so determined satisfy the Adler-Weisberger sum rules for the scattering of pions on any final state

$$\pi_a\, \alpha \to \pi_b\, \beta \ .$$

In a system of mesons consisting of isospin one and zero only, these Adler-Weisberger relations can be expressed in terms of the reduced coupling matrices** $G^{I_\beta I_\alpha}_{\beta\alpha}$ of the pions as

$$\sum_\gamma G^{01}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(\lambda) = 0 \ , \qquad (1.6)$$

$$\sum_\gamma G^{10}_{\beta\gamma}(\lambda)\, G^{01}_{\gamma\alpha}(\lambda) + G^{11}_{\beta\gamma}(\lambda)\, G^{11}_{\gamma\alpha}(\lambda) = 1 \ , \qquad (1.7)$$

where the sum goes over all intermediate states allowed in the s-channel. The informa-

* In Gell-Mann's recent terminology, the first representation denotes the content of constituent quarks in an elementary particle. The mixing establishes the proper amount of current quarks which can be found by probing a particle at infinite momentum with photons and neutrinos [5].

** These matrices $G^{I_\beta I_\alpha}$ are defined by taking the following Clebsch-Gordan coefficients out of $X_a(\lambda)$

$$[X_a(\lambda)]_{\beta\alpha} = \begin{cases} \delta_{\beta a} G^{10} \ , & \text{for } I_\beta = 1, \quad I_\alpha = 0 \ , \\ \delta_{a\alpha} G^{01} \ , & \phantom{\text{for }} I_\beta = 0, \quad I_\alpha = 1 \ , \\ i\epsilon_{\beta a\alpha} G^{11} \ , & \phantom{\text{for }} I_\beta = 1, \quad I_\alpha = 1 \ . \end{cases}$$

Actually these couplings refer to pions continued to zero (mass)^2 *via* the interpolating field $\varphi_\pi = (1/f_\pi m_\pi^2)\, \delta A$. The assumption of PCAC brings us back to the mass shell.

tion contained in the coupling matrices G obtained in such a fashion has so far been used to a rather small extent. Only the absolute magnitude of a few matrix elements $G_{\beta\alpha}^{I_\beta I_\alpha}$ has been tested by comparing with the experimental widths for decays $\beta \to \alpha + \pi$. It is the strength of the Adler-Weisberger relations of being able to determine as well the relative *phases* among the couplings $G_{\beta\alpha}^{I_\beta I_\alpha}$. Other methods which could verify also this aspect of the theory are therefore desirable.

One such method has been developed some time ago. It allows for a calculation of electromagnetic couplings between all particles in the saturation scheme by making a combined use of forward and backward dispersion relations and vector-meson dominance [6]. In this way the anomalous isovector magnetic moment of the Roper resonance was predicted [7] and has since been verified by the phenomenological analysis [8] of photoproduction on deuterons [9]. Furthermore the known multipole couplings of the γ N Δ transition have been correctly reproduced [10].

Recently it has been observed that chiral saturation schemes can also be used to predict the Regge couplings of ρ- and f- trajectories [11]. For this one simply inserts the G-matrices in appropriately chosen finite-energy sum rules. Also this method is sensitive to the relative phase of couplings. The ingredients of this method can briefly be stated as follows.

Let us split the isospin even and odd parts of the scattering amplitude at $t=0$ according to*

$$T_{\beta\alpha}^{(\pm)\mathrm{ba}} = t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu) \pm t_{\beta\alpha}^{(\pm)\mathrm{ba}}(-\nu) \ ,$$

where $t_{\beta\alpha}^{(\pm)\mathrm{ba}}$ contains only the singularities due to the particles exchanged in the s-channel $(\nu = \frac{1}{2}(s-u) = s - \frac{1}{2}(m_\alpha^2 + m_\beta^2))$. According to the ideas of duality [12, 13], $t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu)$ will contain a set of resonance bumps which even out at relatively low energy to form a smooth curve

$$t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu) \underset{\nu\to\infty}{\longrightarrow} (t_{\beta\alpha}^{(\pm)\mathrm{ba}})_{\mathrm{Regge}} = -(C_{\beta\alpha}^{(\pm)\mathrm{ba}}) \frac{\mathrm{e}^{-i\pi\alpha(0)}}{\sin \pi\alpha(0)} \left(\frac{s}{M^2}\right)^{\alpha_{\mathrm{f},\rho}(0)} \tag{1.8}$$

Notice that we have written $(s/M^2)^{\alpha_{\mathrm{f},\rho}(0)}$ rather than the conventional form $(\nu/M^2)^{\alpha_{\mathrm{f},\rho}(0)}$. The reason is that we shall be dealing with rather general scattering amplitudes in which initial and final particles α and β may be highly massive objects. The s-channel singularities in such amplitudes start close to $s=0$ i.e. much below threshold (with one or two pions in the intermediate states). Thus we expect the form $(s/M^2)^{\alpha_{\mathrm{f},\rho}(0)}$ to be relevant down to lower energies than the corresponding version in ν.

With this asymptotic behaviour being known, the difference

$$t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu) - (t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu))_{\mathrm{Regge}} \tag{1.9}$$

* Normalization: $S = 1 - i(2\pi)^4 \delta^4(p_\mathrm{f} - p_\mathrm{i})\, T$, $\quad \langle p'|p\rangle = 2p_0 (2\pi)^3 \delta^3(p'-p)$.

is a superconvergent amplitude and has to satisfy

$$\int_0^\infty \mathrm{d}s \, \mathrm{Im}\,[t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu) - (t_{\beta\alpha}^{(\pm)\mathrm{ba}}(\nu))_{\mathrm{Regge}}] = 0 \; . \tag{1.10}$$

Since the Regge asymptotic form is presumably reached at a rather low energy $s \approx N$ we can truncate the integral and find the finite-energy sum rule [12]:

$$\int_0^N \mathrm{d}s \, \mathrm{Im}\, t_{\beta\alpha}^{(\pm)\mathrm{ba}} \approx -\frac{C^{(\pm)} M^2}{(\alpha_{\mathrm{f},\rho}(0)+1)} \left(\frac{N}{M^2}\right)^{\alpha_{\mathrm{f},\rho}(0)+1} \; . \tag{1.11}$$

The important point now is that the left-hand side can be approximated by the coupling constants $X_{\mathrm{a}}(\lambda)$ calculated in an SU(2) ⊗ SU(2) saturation scheme. For the contribution of the sharp resonances in such a scheme we have

$$\mathrm{Im}\, t_{\beta\alpha}^{(\pm)\mathrm{ba}} = -\pi\left(-\frac{1}{2f_\pi^2}\right) \sum_\gamma \delta(s - m_\gamma^2)(m_\beta^2 - m_\gamma^2)(m_\gamma^2 - m_\alpha^2)\,[X_{\mathrm{b}}(\lambda), X_{\mathrm{a}}(\lambda)]_\pm \; . \tag{1.12}$$

Introducing the matrices

$$(m_\beta^2 - m_\gamma^2)\, X_{\beta\gamma}^{\mathrm{a}}(\lambda) \equiv [m^2, X_{\mathrm{a}}(\lambda)] \equiv -\mathrm{i} m_{\mathrm{a}}^2 \; , \tag{1.13}$$

and performing the integral in (1.11) we obtain

$$\frac{1}{2f_\pi^2}[m_{\mathrm{b}}^2, m_{\mathrm{a}}^2]_\pm \approx -C_{\beta\alpha}^{(\pm)\mathrm{ba}} \frac{(M^2/\pi)}{(\alpha_{\mathrm{f},\rho}(0)+1)} \left(\frac{N}{M^2}\right)^{\alpha_{\mathrm{f},\rho}(0)+1} \; , \tag{1.14}$$

where N is a number somewhat larger than the highest m_α^2 occuring in the saturation scheme.

Notice that due to the absence of exotic Regge trajectories the right-hand side has no $I_t = 2$ part. On the left-hand side, however, there is no *a priori* reason for the $I_t = 2$ part to vanish. The knowledge of its suppression due to the information coming from Regge poles imposes severe restrictions on the interrelation of masses and coupling constants in any scheme. The amount of suppression should become more effective with growing saturation schemes while being hardly noticable if only a few particle couplings are available. We shall come back to this point in the next section.

In an earlier paper another finite-energy sum rule has been derived involving

$$\delta_{\mathrm{ab}}\, m_4^2 = [X_{\mathrm{a}}(\lambda), [X_{\mathrm{b}}(\lambda), m^2]] \; . \tag{1.15}$$

This was done by considering the finite-contour dispersion relation

$$T_{\beta\alpha}^{(+)\mathrm{ba}}(\nu_{\mathrm{th}}^2) = \frac{1}{\pi} \int_0^N \frac{\mathrm{Im}\, t^{(+)}(\nu')\,\mathrm{d}\nu'^2}{\nu'^2 - \nu_{\mathrm{th}}^2} + \oint \frac{\mathrm{d}\nu'^2\, C^{(+)}(s/M^2)^{\alpha_{\mathrm{f}}(0)}}{\nu'^2 - \nu_{\mathrm{th}}^2} \; , \tag{1.16}$$

and using the low-energy theorem

$$T^{(+)\mathrm{ba}}_{\beta\alpha}(\nu^2_{\mathrm{th}}) = \Sigma^{\mathrm{ba}}_{\beta\alpha}/f^2_\pi \ , \tag{1.17}$$

where $\Sigma^{\mathrm{ba}}_{\beta\alpha}$ denotes the matrix elements of the Σ-commutator

$$\Sigma^{\mathrm{ba}} = \tfrac{1}{2} i\,([Q^{\mathrm{a}}, \partial A^{\mathrm{b}}] + [Q^{\mathrm{b}}, \partial A^{\mathrm{a}}]) \ , \tag{1.18}$$

and ν^2_{th} stands for the threshold value $\nu^2_{\mathrm{th}} = [\frac{1}{2}(m^2_\alpha - m^2_\beta)]^2$ of ν^2. Inserting the coupling matrices $X^{\mathrm{a}}_{\beta\alpha}(\lambda)$ into the first integral in (1.16) we find

$$m^2_4 = \Sigma - \frac{2f^2_\pi}{\pi}\, C^{(+)} \frac{N^{\alpha_{\mathrm{f}}(0)}}{\alpha_{\mathrm{f}}(0)} \left[1 - E\left(\frac{m^2_\alpha}{N}, \frac{m^2_\beta}{N}\right)\right] . \tag{1.19}$$

Here the usual Regge expression is modified by a factor $(1 - E)$ where E is a correction term having its origin in the denominator $(\nu'^2 - \nu^2_{\mathrm{th}})$ in the last integral of (1.16) and tends to zero for (m^2_α/N), $(m^2_\beta/N) \to 0$. The correction term can therefore be safely neglected in large saturation schemes when the pions are scattered on particles with low masses. Explicitly this term is

$$E\left(\frac{m^2_\alpha}{N}, \frac{m^2_\beta}{N}\right) = 1 - \frac{1}{2} \sum_{n=0}^{\infty} \frac{\alpha(0)}{\alpha(0) - n} \left[\left(\frac{m^2_\alpha}{N}\right)^n + \left(\frac{m^2_\beta}{N}\right)^n\right] . \tag{1.20}$$

Since the intercept $\alpha(0)$ is about $\frac{1}{2}$ the summation can easily be carried out:

$$E\left(\frac{m^2_\alpha}{N}, \frac{m^2_\beta}{N}\right) = + \sqrt{\frac{m^2_\alpha}{N}}\, \ln\left[\frac{1 + \sqrt{m^2_\alpha/N}}{1 - \sqrt{m^2_\alpha/N}}\right] + \sqrt{\frac{m^2_\beta}{N}}\, \ln\left[\frac{1 + \sqrt{m^2_\beta/N}}{1 - \sqrt{m^2_\beta/N}}\right] . \tag{1.21}$$

If N is rather small E may take such large values that $(1 - E)$ can be close to zero or even become negative. We believe that in such cases the sum rules are quite unreliable since they depend on the validity of the Regge asymptotic form all the way down to threshold $\nu^2 \approx \nu^2_{\mathrm{th}}$, (i.e. $s \equiv s_{\mathrm{th}} = \max(m^2_\alpha, m^2_\beta)$). This situation occurs, for example, if one considers $\pi A_1 \to \pi A_1$ scattering in a saturation scheme involving only π, ρ, A_1, σ. Then the only intermediate resonances ρ, σ lie *below* the threshold energy $s = m^2_{A_1}$, and the Regge asymptotic form will hardly have been reached at $N \approx m^2_{A_1}$.

It is the purpose of this paper to employ the largest meson saturation scheme available in the literature in order to calculate Reggeized f- and ρ- couplings from the sum rules (1.14). Many of these couplings will hopefully be tested in the future when good phenomenological fits to production amplitudes for mesons in pion and photon collisions (using vector-meson dominance to relate to the ρ-meson) will become available. A few couplings are compared with the data available at present and with predictions of competing models.

2. The meson saturation scheme

Let us consider explicitly the mesons contained in the representations **15**, $L = 0$, and **15**, $L = 1$ of SU(4) $\otimes$ O_3 (ref. [3]). For a brief orientation on their properties we have given a Chew-Frautschi plot of these mesons in fig. 1, with the parity (P) and G-parity quantum numbers added on as superscripts. The states of $L = 0$ can be decomposed with respect to the chiral subgroup (1.4) and (1.5) as

$$\rho^\lambda = (v^1, \bar{t}, v^{-1}), \qquad \omega^\lambda = (v_4^1, s^0, -v_4^{-1}), \qquad \pi = t^{(0)} , \tag{2.1}$$

where λ denotes the helicity and $s, v, \bar{t}, t$ are* the standard abbreviations to the (0,0), $(\frac{1}{2},\frac{1}{2})$, [(1,0) + (0,1)] and [(1,0) − (0,1)] representations of SU(2) $\otimes$ SU(2) respectively.

In combining these states with orbital $L = 1$, one finds the chiral contents: **

$\lambda=2$	$\lambda=1$	$\lambda=0$	GP
$A_2 = v'$	$A_2 = \frac{1}{\sqrt{2}}(v' + \bar{t}\,')$	$A_2 = \frac{1}{\sqrt{3}}[v_1 + \sqrt{2}\,\bar{t}\,']$	− +
$f = v_4'$	$f = \frac{1}{\sqrt{2}}(v_4' + s)$	$f = \frac{1}{\sqrt{3}}[v_4' + \sqrt{2}\,s]$	+ +
	$A_1 = \frac{1}{\sqrt{2}}(v' - \bar{t}\,')$	$A_1 = v_1'$	− +
	$D = \frac{1}{\sqrt{2}}(v_4' - s)$	$D = v_4$	+ +
	$B = t'$	$B = t'$	+ +
		$A_0 = \frac{1}{\sqrt{3}}[\sqrt{2}\,v_1 - \bar{t}\,']$	− +
		$\sigma = \frac{1}{\sqrt{3}}[\sqrt{2}\,v_4' - s]$	+ +

Representation mixing has to be performed at every helicity separately. It has to respect the two important properties of $X(\lambda)$:

* The notation makes use of the four-vector language of the group O(4) which is isomorphic to SU(2) $\otimes$ SU(2). See ref. [2].

** For brevity we have dropped the orbital states in the notation since their magnetic quantum numbers are uniquely determined by the helicities. However, we have to remember that $t, \bar{t}$ and v at $L = 1$ are different from those at $L = 0$. We have indicated this fact by using primes on these quantities for $L = 1$. At $\lambda = 0$ we have also used the more concise notation $\sqrt{2}\,v_1 = (v^1 + v^{-1})$, $\sqrt{2} v_4 = (v_4^1 + v_4^{-1})$, $\sqrt{2} v_1' = (v^1 - v^{-1})$ and $\sqrt{2} v_4' = (v_4^1 - v_4^{-1})$ since only the chiral properties are of interest here.

$$G\,X(\lambda)\,G^{-1} = -X(\lambda)\ , \qquad X_{\beta\alpha}(-\lambda) = -\eta_\beta\eta_\alpha X_{\beta\alpha}(\lambda)\ , \tag{2.2}$$

where G is the G-parity operator, and η_α = intrinsic parity $\times\,(-1)^{J_\alpha}$ denotes the normality of the particles.

Inspection of the states shows that one can mix only the π with the A_1 meson at helicity $\lambda = 0$ and the ρ with the B-meson at $\lambda = 1$, i.e.,

$$\pi = \cos\psi\ t + \sin\psi\ v_1'\ , \qquad A_1 = -\sin\psi\ t + \cos\psi\ v_1'\ , \tag{2.3}$$

$$\rho^{(1)} = \cos\phi\, v' + \sin\phi\, t'\ , \qquad B^{(1)} = -\sin\phi\, v' + \cos\phi\, t'\ . \tag{2.4}$$

From the standpoint of SU(2) ⊗ SU(2) the mixing angles ψ and ϕ are completely arbitrary. The resulting G-matrices are obtained for helicity zero as

$$G^{01}_{(0)} = \begin{array}{c} \\ \sigma \\ \mathrm{f} \\ \mathrm{D} \end{array} \begin{array}{c} \begin{array}{cccccc} \pi & A_1 & \rho & A_2 & B & A_0 \end{array} \\ \left[\begin{array}{cccccc} \sqrt{\tfrac{2}{3}}\sin\psi & \sqrt{\tfrac{2}{3}}\cos\psi & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}}\sin\psi & \frac{1}{\sqrt{3}}\cos\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & \sqrt{\tfrac{2}{3}} \end{array}\right] \end{array} \tag{2.5}$$

$$G^{11}(0) = \begin{array}{c} \\ \pi \\ A_1 \\ \rho \\ A_2 \\ B \\ A_0 \end{array} \begin{array}{c} \begin{array}{cccccc} \pi & A_1 & \rho & A_2 & B & A_0 \end{array} \\ \left[\begin{array}{cccccc} 0 & 0 & \cos\psi & & & \\ 0 & 0 & -\sin\psi & & 0 & \\ \cos\psi & -\sin\psi & 0 & & & \\ & & & 0 & \sqrt{\tfrac{2}{3}} & 0 \\ & 0 & & \sqrt{\tfrac{2}{3}} & 0 & -\frac{1}{\sqrt{3}} \\ & & & 0 & -\frac{1}{\sqrt{3}} & 0 \end{array}\right] \end{array}\ , \tag{2.6}$$

for helicity one as

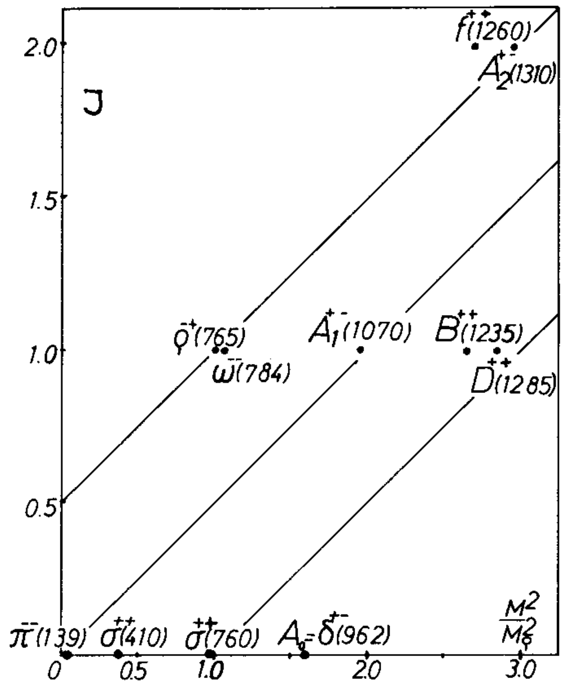

Fig. 1. A Chew-Frautschi plot of particles present in our model. The parity and G-parity quantum numbers are given as superscripts, in that order.

$$G^{01}(1) = \begin{array}{c} \\ f \\ D \\ \omega \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & \rho & B & A_2 \end{array} \\ \left[\begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \cos\phi & -\sin\phi & 0 \end{array}\right] \end{array}, \qquad (2.7)$$

$$G^{11}(1) = \begin{array}{c} \\ A_1 \\ \rho \\ B \\ A_2 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & \rho & B & A_2 \end{array} \\ \left[\begin{array}{cccc} 0 & -\frac{1}{\sqrt{2}}\sin\phi & -\frac{1}{\sqrt{2}}\cos\phi & 0 \\ -\frac{1}{\sqrt{2}}\sin\phi & 0 & 0 & \frac{1}{\sqrt{2}}\sin\phi \\ -\frac{1}{\sqrt{2}}\cos\phi & 0 & 0 & \frac{1}{\sqrt{2}}\cos\phi \\ 0 & \frac{1}{\sqrt{2}}\sin\phi & \frac{1}{\sqrt{2}}\cos\phi & 0 \end{array}\right] \end{array}, \qquad (2.8)$$

and for helicity two as

$$G^{01}(2) = G_{\mathrm{fA}_2}(2) = 1 \;, \tag{2.9}$$

$$G^{11}(2) = G_{\mathrm{A}_2\mathrm{A}_2}(2) = 0 \;. \tag{2.10}$$

It is easy to check that these matrices do indeed fulfill the Adler-Weisberger relations (1.6) and (1.7).

Due to the restriction to a finite set of resonances the determination of coupling constants will involve errors that are hard to assess. If one considers the pions on two-low-lying particles, most of the important resonances necessary for a saturation of the Adler-Weisberger relations will be contained in our scheme. If, on the other hand, initial and final particles are rather heavy, our scheme has too few resonances to offer above these particles to provide for an appropriate saturation of the sum rules. The scheme will exaggerate the couplings in order to make up for the absence of the higher resonances*. An example is $\pi A_2 \to \pi A_2$ scattering at $\lambda = 2$. This amplitude is saturated with just the f-meson yielding $G^{10}_{\mathrm{A}_2\mathrm{f}}(0) = 1$, which is probably a poor approximation. In general one should expect that the larger number of particles present in an Adler-Weisberger relation above the target and final mass, the more reliable are the coupling constants determined from it.

The resulting coupling can be compared with experiment by calculating the decay width for $A_1 \to \rho\pi$, $A_2 \to \rho\pi$, $B \to \omega\pi$, $\sigma \to \pi\pi$, $f \to \pi\pi$. One finds that mixing angles $\sin\psi = 1/\sqrt{3}$ and $\sin\phi = 1/\sqrt{6}$ lead to reasonable overall fits of the existing data (for the details see appendix A).

3. The Regge couplings

The calculation is conveniently performed in terms of the reduced matrix elements $M^{I_\beta I_\alpha}_{\mathrm{v}\beta\alpha}$ of $(m^2_{\mathrm{a}})_{\beta\alpha}$. These are defined in complete analogy with $G^{I_\beta I_\alpha}_{\beta\alpha}$. Since by definition $m^2_{\mathrm{a}} = +i\,[m^2, X(\lambda)]$ we have

$$M^{I_\beta I_\alpha}_{\mathrm{v}\beta\alpha} = +i(m^2_\beta - m^2_\alpha)\, G^{I_\beta I_\alpha}_{\beta\alpha} \;. \tag{3.1}$$

The commutator $[m^2_{\mathrm{b}}, m^2_{\mathrm{a}}]$ consists only of an isospin one operator r_{c}:

$$[m^2_{\mathrm{b}}, m^2_{\mathrm{a}}] \equiv i\,\epsilon_{\mathrm{bac}}\, r_{\mathrm{c}} \;. \tag{3.2}$$

Its reduced matrix elements are

$$R^{10}_{\beta\alpha} = 2\,\sum_\gamma M^{11}_{\mathrm{v}\beta\alpha}\, M^{10}_{\mathrm{v}\gamma\alpha} \;, \tag{3.2}$$

* Due to the positive definiteness of the resonance contributions in all elastic sum rules.

$$R^{11}_{\beta\alpha} = \sum_{\gamma} (M^{10}_{v\beta\gamma} M^{01}_{v\beta\gamma} + M^{11}_{\beta\gamma} M^{11}_{\gamma\alpha}) \ . \tag{3.4}$$

The anticommutator $\{m_b^2, m_a^2\}$, on the other hand, can be written as

$$\{m_b^2, m_a^2\} = \delta_{ba} f + e_{ba} \ , \tag{3.5}$$

where the traceless tensor e_{ba} is an exotic isospin 2 operator while f is of pure isospin zero.

The reduced matrix elements of f are *

$$F^{00}_{\beta\alpha} = 2\, M^{01}_{v\beta\gamma} M^{10}_{v\gamma\alpha} \ , \qquad F^{11}_{\beta\alpha} = \tfrac{2}{3}(M^{10}_{v\beta\gamma} M^{01}_{v\gamma\alpha} + 2 M^{11}_{v\beta\gamma} M^{11}_{v\gamma\alpha}) \ . \tag{3.6}$$

The exotic operator can only contribute between isospin one states. There it yields**

$$E^{11}_{\beta\alpha} = \tfrac{2}{3}(M^{10}_v M^{01}_v - M^{11}_v M^{11}_v)_{\beta\alpha} \ . \tag{3.7}$$

The results are displayed on table 1 for helicity zero and on table 2 for helicity one scattering, with particle symbols denoting particle mass squared.

The Regge contributions of f- and ρ- mesons to the amplitudes

$$T^{(+)} \equiv \tfrac{1}{3}\left(T^{I_t=0} + T^{I_t=2}\right)^{ba}_{\beta\alpha} \ , \qquad T^{(-)} \equiv \tfrac{1}{2}\left(T^{I_t=1}\right)^{ba}_{\beta\alpha} \ ,$$

are obtained from the numbers in the tables by using (1.14):

$$C^{(\pm)}_{\beta\alpha} = -\frac{\pi}{2f_\pi^2} \frac{(\alpha_{f,\rho}(0)+1)}{M^2} \left(\frac{M^2}{N}\right)^{\alpha_{f,\rho}(0)+1} \times \begin{Bmatrix} F_{\beta\alpha} \\ R_{\beta\alpha} \end{Bmatrix} \ . \tag{3.8}$$

The commonly accepted values for the intercepts of the f- and ρ- trajectories are $\alpha_f(0) \approx \alpha_\rho(0) \approx \frac{1}{2}$. The energy N has to be chosen somewhere above the highest

* For isospin zero operators we go to reduced matrix elements simply by removing the isospin conserving $\delta_{\beta\alpha}$.

** Notice that between isospin one states

$$\delta_{ba} f^{11}_{\beta\alpha} = \delta_{ba}\, \delta_{\beta\alpha} F^{11}_{\beta\alpha} = 2(P^{I_t=0}) F^{11}_{\beta\alpha} \ ,$$

correspondingly we have defined

$$(e_{ba})^{11}_{\beta\alpha} = [(\delta_{b\beta}\delta_{a\alpha} + \delta_{b\alpha}\delta_{a\beta}) - \tfrac{2}{3}\delta_{ba}\delta_{\beta\alpha}]\, E^{11}_{\beta\alpha} \equiv 2P^{I_t=2} E^{11}_{\beta\alpha} \ .$$

This definition is convenient since it makes R and F, E contribute directly to $T^{(-)} = \frac{1}{2} T^{I_t=1}$ and $T^{(+)} = \frac{1}{3}(T^{I_t=0} + T^{I_t=2})$.

Table 1
For the processes $\pi\alpha \to \pi\beta$ the Adler-Weisberger sum rules are displayed together with the finite energy sum rules for the I_t = 0,2 and 1 amplitudes which are denoted F, E and R, respectively. Helicity $\lambda = 0$, $s = \sqrt{\frac{1}{3}}$, $c = \sqrt{\frac{2}{3}}$.

Reaction $\pi\beta \to \pi\alpha$	Adler-Weisberger relation	Matrix elements of $\begin{pmatrix} \frac{3}{2}F \\ \frac{3}{2}E \\ R \end{pmatrix}$		
a) $I_\beta = I_\alpha = 1$				
$\pi\pi \to \pi\pi$	$\underset{\sigma}{\tfrac{1}{3}s^2 (2}+\underset{f}{1)}+\underset{\rho}{c^2} = 1$	$\frac{1}{3}s^2\,[2(\pi-\sigma)^2+(\pi-\mathrm{f})^2]+\begin{pmatrix}2\\-1\\1\end{pmatrix}c^2\,(\pi-\rho)^2$	$=\begin{pmatrix}\frac{3}{2}(1.50)\\ \frac{3}{2}(0.25)\\ 1.63\end{pmatrix}$	ρ^2
$\pi A_1 \to \pi A_1$	$\underset{\sigma}{\tfrac{1}{3}c^2 (2}+\underset{f}{1)}+\underset{\rho}{s^2} = 1$	$\frac{1}{3}c^2\,[2(A_1-\sigma)^2+(A_1-\mathrm{f})^2]+\begin{pmatrix}2\\-1\\1\end{pmatrix}s^2\,(A_1-\rho)^2$	$=\begin{pmatrix}\frac{3}{2}(0.78)\\ \frac{3}{2}(0.16)\\ 0.87\end{pmatrix}$	ρ^2
$\pi\pi \to \pi A_1$	$\underset{\sigma}{\tfrac{1}{3}sc\,(2}+\underset{f}{1)}-\underset{\rho}{3} = 0$	$\frac{1}{3}sc\,[2(\pi-\sigma)(A_1-\sigma)+(\pi-\mathrm{f})(A_1-\mathrm{f})+\begin{pmatrix}2\\-1\\1\end{pmatrix}3]$	$=\begin{pmatrix}\frac{3}{2}(\ 0.60)\\ \frac{3}{2}(-0.31)\\ 0.45\end{pmatrix}$	ρ^2
$\pi A_2 \to \pi A_2$	$\underset{D}{\tfrac{1}{3}}+\underset{B}{\tfrac{2}{3}} = 1$	$\frac{1}{3}[(A_2-D)^2+\begin{pmatrix}2\\-1\\1\end{pmatrix}2(A_2-B)^2]$	$=\begin{pmatrix}\frac{3}{2}(\ 0.08)\\ \frac{3}{2}(-0.04)\\ 0.06\end{pmatrix}$	ρ^2
$\pi A_0 \to \pi A_0$	$\underset{D}{\tfrac{2}{3}}+\underset{B}{\tfrac{1}{3}} = 1$	$\frac{1}{3}[2(A_0-D)^2+\begin{pmatrix}2\\-1\\1\end{pmatrix}(A_0-B)^2]$	$=\begin{pmatrix}\frac{3}{2}(\ 1.08)\\ \frac{3}{2}(\ 0.42)\\ 1.30\end{pmatrix}$	ρ^2
$\pi A_2 \to \pi A_0$	$\underset{D}{\tfrac{1}{3}}-\underset{B}{\tfrac{2}{3}} = 0$	$\frac{1}{3}\sqrt{2}\,[(A_2-D)(A_0-D)-\begin{pmatrix}2\\-1\\1\end{pmatrix}(A_2-B)\,(A_0-B)]$	$=\begin{pmatrix}\frac{3}{2}(\ 0.15)\\ \frac{3}{2}(-0.13)\\ 0.09\end{pmatrix}$	ρ^2
$\pi B \to \pi B$	$\underset{A_2}{\tfrac{2}{3}}+\underset{A_0}{\tfrac{1}{3}} = 1$	$\frac{1}{3}\begin{pmatrix}2\\-1\\1\end{pmatrix}[2(B-A_2)^2+(B-A_0)^2]$	$=\begin{pmatrix}\frac{3}{2}(\ 0.52)\\ \frac{3}{2}(-0.26)\\ 0.39\end{pmatrix}$	ρ^2
$\pi\rho \to \pi\rho$	$\underset{\pi}{c^2}+\underset{A_1}{s^2} = 1$	$\begin{pmatrix}2\\-1\\1\end{pmatrix}[c^2\,(\rho-\pi)^2+s^2\,(\rho-A_1)^2]$	$=\begin{pmatrix}\frac{3}{2}(\ 1.28)\\ \frac{3}{2}(-0.64)\\ 0.96\end{pmatrix}$	ρ^2
b) $I_\beta = 0, I_\alpha = 1$		Matrix elements of R		
$\pi\sigma \to \pi\rho$	$2\sqrt{\tfrac{2}{3}}\,sc\,(\underset{\pi}{1}-\underset{A_1}{1}) = 0$	$2\sqrt{\frac{2}{3}}\,sc\,[(\sigma-\pi)(\rho-\pi)-(\sigma-A_1)(\rho-A_1)]\ \approx$	-0.06	ρ^2
$\pi f \to \pi\rho$	$2\sqrt{\tfrac{1}{3}}\,sc\,(\underset{\pi}{1}-\underset{A_1}{1}) = 0$	$2\sqrt{\frac{1}{3}}\,sc\,[(\mathrm{f}-\pi)(\rho-\pi)-(\mathrm{f}-A_1)(\rho-A_1)]\ \approx$	$+1.80$	ρ^2
$\pi D \to \pi B$	$2\sqrt{\tfrac{2}{3}}\,sc\,(\underset{A_2}{1}-\underset{A_0}{1}) = 0$	$2\sqrt{\frac{2}{3}}\,sc\,[(B-A_2)(D-A_2)-(B-A_0)(D-A_0)]\ \approx$	-1.10	ρ^2

Table 1 (continued).

Reaction	Adler-Weisberger relation	Matrix elements of $\begin{Bmatrix} \frac{3}{2}F \\ \frac{3}{2}E \\ R \end{Bmatrix}$			
c) $I_\beta = I_\alpha = 0$		Matrix elements of F			
$\pi\sigma \to \pi\sigma$	0 = 0	$\frac{4}{3}[s^2\,(\sigma-\pi)^2 + c^2\,(\sigma - A_1)^2]$	$\approx$	1.30	ρ^2
$\pi f \to \pi f$		$\frac{2}{3}[s^2\,(f-\pi)^2 + c^2\,(f-A_1)^2]$	$\approx$	1.80	ρ^2
$\pi\sigma \to \pi f$		$\frac{2}{3}\sqrt{2}\,[s^2\,(\sigma-\pi)\,(f-\pi) + c^2\,(\sigma-A_1)\,(f-A_1)]$	$\approx$	0.37	ρ^2
$\pi D \to \pi D$		$\frac{2}{3}[(D-A_2)^2 + 2(D-A_0)^2]$	$\approx$	1.92	ρ^2

The common helicity of target and final particles is $\lambda = 0$. The particle symbols denote the particle mass squared.

mass squared. Let us take $N \approx 4.5\, m_\rho^2$. Choosing $M^2 \approx 2m_\rho^2 = 2\rho \approx 1$ MeV^2 we finally obtain

$$C_{\beta\alpha}^{(\pm)} \approx -22.7 \begin{Bmatrix} F \\ R \end{Bmatrix} \frac{1}{\rho^2} \,. \tag{3.9}$$

Thus multiplication of our values in tables 1 and 2 by -22.7 yields directly the desired Regge couplings. Notice that the overall size of the couplings is dependent on the value chosen for N. Had we used $N \approx 5\rho$ we would have

$$C_{\beta\alpha}^{(\pm)} \approx -19.4 \begin{Bmatrix} F \\ R \end{Bmatrix} \frac{1}{\rho^2} \,.$$

This discrepancy indicates the typical systematic error inherent in our predictions. For the sake of simplicity let us use the factor -20 in translating F- and R- values to $C^{(\pm)}$.

We wish to bring attention to the approximate exchange degeneracy [14] $F \approx R$ predicted by our scheme. It is known from general considerations on duality that exchange degeneracy goes together with the absence of exotic amplitudes at high energy. In our algebraic scheme this connection is manifest. From eqs. (3.3), (3.5) and (3.6) we have

$$R - F = \tfrac{1}{2}E \,.$$

A perusal of tables 1 and 2 shows us that E is in fact considerably suppressed relative to F. The reason is roughly that in F one is dealing with a sum of the isospin zero

Table 2

For the processes $\pi\alpha \to \pi\beta$ the Adler–Weisberger sum rules are displayed together with the finite energy sum rules for the $I_t = 0, 2$ and 1 amplitudes which are denoted by F, E and R, respectively. Helicity $\lambda = 1$, $s = \sqrt{\frac{1}{6}}$, $c = \sqrt{\frac{5}{6}}$.

Reaction $\pi\beta \to \pi\alpha$	Adler-Weisberger relation		Matrix elements of $\begin{pmatrix} \frac{3}{2}E \\ \frac{3}{2}E \\ R \end{pmatrix}$		
a) $I_\beta = I_\alpha = 1$					
$\pi\rho \to \pi\rho$	$\underset{\omega}{c^2} + \frac{1}{2}s^2(\underset{A_1}{1} + \underset{A_2}{1})$	$= 1$	$c^2(\rho-\omega)^2 + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} s^2 [(\rho - A_1)^2 + (\rho - A_2)^2]$	$= \begin{bmatrix} \frac{3}{2}(\ 0.51) \\ \frac{3}{2}(-0.26) \\ 0.38 \end{bmatrix}$	ρ^2
$\pi A_1 \to \pi A_1$	$\frac{1}{4}(\underset{f}{1} + \underset{D}{1}) + \frac{1}{2}(\underset{\rho}{s^2} + \underset{B}{c^2})$	$= 1$	$\frac{1}{4}[(A_1 - f)^2 + (A_1 - D)^2] + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2}[s^2(A_1 - \rho)^2 + c^2(A_1 - B)^2]$	$= \begin{bmatrix} \frac{3}{2}(\ 0.50) \\ \frac{3}{2}(\ 0.03) \\ 0.51 \end{bmatrix}$	ρ^2
$\pi A_2 \to \pi A_2$	$\frac{1}{4}(\underset{f}{1} + \underset{D}{1}) + \frac{1}{2}(\underset{\rho}{s^2} + \underset{B}{c^2})$	$= 1$	$\frac{1}{4}[(A_2 - f)^2 + (A_2 - D)^2] + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2}[s^2(A_2 - \rho)^2 + c^2(A_1 - B)^2]$	$= \begin{bmatrix} \frac{3}{2}(\ 0.46) \\ \frac{3}{2}(-0.22) \\ 0.35 \end{bmatrix}$	ρ^2
$\pi A_1 \to \pi A_2$	$\frac{1}{4}(\underset{f}{1} + \underset{D}{1}) - \frac{1}{2}(\underset{\rho}{s^2} + \underset{B}{c^2})$	$= 0$	$\begin{cases} \frac{1}{4}[(A_1 - f)(A_2 - f) + (A_1 - D)(A_2 - D)] \\ - \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2}[s^2(A_1 - \rho)(A_2 - \rho) + c^2(A_1 - B)(A_2 - B)] \end{cases}$	$= \begin{bmatrix} \frac{3}{2}(-0.15) \\ \frac{3}{2}(\ 0.02) \\ -0.14 \end{bmatrix}$	ρ^2
$\pi B \to \pi B$	$\underset{\omega}{s^2} + \frac{1}{2}c^2(\underset{A_1}{1} + \underset{A_2}{1})$	$= 1$	$s^2(B - \omega)^2 + \frac{1}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} c^2 [(B - A_1)^2 + (B - A_2)^2]$	$= \begin{bmatrix} \frac{3}{2}(\ 0.53) \\ \frac{3}{2}(\ 0.16) \\ 0.61 \end{bmatrix}$	ρ^2
$\pi B \to \pi\rho$	$sc(-\underset{\omega}{1} + \underset{A_1}{\frac{1}{2}} + \underset{A_2}{\frac{1}{2}})$	$= 0$	$sc\left\{-(B - \omega)(\rho - \omega) + \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \frac{1}{2}[(B - A_1)(\rho - A_2) + (B - A_2)(\rho - A_2)]\right.$	$\approx \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	

Table 2 (continued).

Reaction $\pi\beta \to \pi\alpha$	Adler-Weisberger relation		Matrix elements of $\begin{Bmatrix} \frac{3}{2}E \\ \frac{3}{2}E \\ R \end{Bmatrix}$		
b) $I_\beta = 0, I_\alpha = 1$			Matrix elements of R:		
$\pi f \to \pi\rho$	$\frac{s}{\sqrt{2}}(\underset{A_1}{-1}\;\underset{A_2}{+1})$	$=0$	$(s/\sqrt{2})\,[-(f-A_1)(\rho-A_1)+(f-A_2)(\rho-A_2)]$	$= +0.31$	ρ^2
$\pi f \to \pi B$	$\frac{c}{\sqrt{2}}(\underset{A_1}{-1}\;\underset{A_2}{+1})$	$=0$	$(c/\sqrt{2})\,[-(f-A_1)(B-A_1)+(f-A_2)(B-A_2)]$	$= -0.23$	ρ^2
$\pi D \to \pi\rho$	$\frac{s}{\sqrt{2}}(\underset{A_1}{-1}\;\underset{A_2}{+1})$	$=0$	$(s/\sqrt{2})\,[-(D-A_1)(\rho-A_1)+(D-A_1)(\rho-A_2)]$	$\approx +0.35$	ρ^2
$\pi D \to \pi B$	$\frac{c}{\sqrt{2}}(\underset{A_1}{-1}\;\underset{A_2}{+1})$	$=0$	$(c/\sqrt{2})\,[-(D-A_1)(B-A_1)+(D-A_2)(B-A_2)]$	≈ -0.28	ρ^2
$\pi\omega \to \pi A_1$	$\sqrt{2}\,cs\,(\underset{\rho}{-1}\;\underset{B}{+1})$	$=0$	$(\sqrt{2}\,sc)\,[-(\omega-\rho)(A_1-\rho)+(\omega-B)(A_1-B)]$	$\approx +0.30$	ρ^2
$\pi\omega \to \pi A_2$	$\sqrt{2}\,cs\,(\underset{\rho}{1}\;\underset{B}{-1})$	$=0$	$(\sqrt{2}\,sc)\,[+(\omega-\rho)(A_2-\rho)-(\omega-B)(A_2-B)]$	≈ 0.30	ρ^2

Table 2 (continued).

Reaction		Adler-Weisberger relation	Matrix elements of $\begin{pmatrix} \frac{3}{2}E \\ \frac{3}{2}E \\ R \end{pmatrix}$	
(c)			Matrix elements of F:	
$I_\beta = 0 = I_\alpha$				
$\pi f \to \pi f$			$\frac{1}{2}[(f-A_1)^2 + (f-A_2)^2]$	$\approx 0.26\ \rho^2$
$\pi D \to \pi D$		$0 = 0$	$\frac{1}{2}[(D-A_1)^2 + (D-A_2)^2]$	$\approx 0.33\ \rho^2$
$\pi f \to \pi D$			$\frac{1}{2}[(f-A_1)(D-A_1) + (f-A_2)(D-A_2)]$	$\approx 0.29\ \rho^2$
$\pi\omega \to \pi\omega$			$2[c^2(\omega-\rho)^2 + s^2(\omega-B)^2]$	$\approx 0.85\ \rho^2$

The common helicity of target and final particles is $\lambda = 1$. The particle symbols denote the particle mass squared.

and (twice) the isospin one contributions, while in E the two contributions are of opposite sign. If an amplitude has only isospin one intermediate states, the suppression of exotics is only by a factor $-\frac{1}{2}$ as, for example, in $\pi\rho \to \pi\rho$ and $\pi B \to \pi B$ at helicity $\lambda = 0$ (see table 1). In the first case this is a direct consequence of the fact that ω cannot contribute due to normality selection rules and that no meson with $I^G = 0^-$ and normality -1 is known to couple significantly to $\pi\rho$ ($\phi(1019)$ has only $\Gamma_{\phi\to\rho\pi} \leqslant 1$ MeV, the next candidate being ϕ_N (1655) with $\Gamma_{\phi_N\to\rho\pi} \leqslant 140$ MeV). In the second case the decoupling of ω (at helicity $\lambda = 0$) is a prediction of the model which, however, agrees with experiment (see appendix A). Also here no other $I^G = 0^-$ meson of normality -1 is known to couple to $B\pi$. We conclude that forward $\pi\rho$ and πB scattering at $\lambda = 0$ should show some exotic amplitude and that there exchange degeneracy should be very approximate. The success in obtaining a suppression of the exotic amplitudes may be related to the fact that our particle multiplets are taken from the quark model. We wish to mention that considerable effort has been devoted to establishing quark-like SU(6) × O(3) multiplet structures purely from Regge considerations without much success [15]. There one assumes all baryons to lie on completely degenerate Regge trajectories and evaluates the consequences of the condition that exotic amplitudes vanish at high energies. The multiplets found in such calculations are quite different from those expected from the quark model. The connection between the two approaches is certainly worth further investigation.

Before we go on to discuss the Regge couplings we find it interesting to compare the results with what has been obtained previously in a small saturation scheme containing only π, ρ, A_1 and σ mesons. Our formula (3.8) indicates that by using larger and larger saturation schemes the matrix elements of F and R should increase with a power $\approx N^{\frac{3}{2}}$. Consider π-π scattering for which we may expect our saturation scheme to work best*. The small scheme (constructed for zero helicity) yielded

$$[R]_{\pi\pi} = \rho^2 \ , \tag{3.10}$$

while the larger scheme gives (table 1):

$$R_{\pi\pi} = 1.63\, \rho^2 \tag{3.11}$$

The increase by a factor 1.63 is just what one expects from the behaviour $N^{\frac{3}{2}}$ since N has meanwhile increased from $\approx 3\, m_\rho^2$ in the previous scheme to $\approx 4.5\, m_\rho^2$ in the present**. Note that if one would compare two large adjacent saturation schemes one could establish accurately the intercepts $\alpha(0)$ of the leading Regge trajectories.

It is worth emphasizing that the scheme with π, ρ, A_1 and σ mesons is not very suitable for a calculation of Regge couplings within our framework, even though it has some interesting features [11]. For instance, it leads to ρ -universality in the form

$$R_{\pi\pi} = R_{A_1A_1} = R_{\rho\rho} = \rho^2. \tag{3.12}$$

However, it has rather large exotic parts in the even amplitude and exchange degeneracy is no longer valid for all amplitudes:

$$F_{\pi\pi} = F_{A_1A_1} = \rho^2 \ , \qquad F_{\rho\rho} = 4F_{\pi A_1} = \tfrac{4}{3}\rho^2 \ , \tag{3.13}$$

$$E_{\rho\rho} = -E_{\pi A_1} = \tfrac{2}{3}\rho^2 \ . \tag{3.14}$$

Curiously enough, despite large isospin 2 parts in the matrix elements of $\{m_b^2, m_a^2\}$, the super-convergence relation

$$T^{(+)}(\nu_{\rm th})\Big|_{I=2} = \frac{\Sigma(I=2)}{f_\pi^2} = 0 = \frac{1}{\pi}\int_0^\infty \frac{d\nu'^2 \, {\rm Im}\, T^{(+)}_{I=2}(\nu')}{\nu'^2 - \nu_{\rm th}^2} \ , \tag{3.15}$$

leading to [2]

$$\Big\{[X_b, [X_a, m^2]] + (b \Leftrightarrow a)\Big\}_{I=2} = 0 \ , \tag{3.16}$$

and the superconvergence relation

$$\frac{1}{\pi}\int_0^\infty d\nu'^2 \, {\rm Im}\, T^{(+)}_{I=2}(\nu') = 0 \ , \tag{3.17}$$

* Since the external particle masses are lowest.

** Naturally complete continuity cannot be expected in going from such a small scheme containing four mesons only to the present larger scheme.

giving

$$\left\{[[X_b, m^2]\ ,\ [[X_a, m^2], m^2],] + (b \Leftrightarrow a)\right\}_{I=2} = 0\ , \tag{3.18}$$

are satisfied [16] for the same choice of the mixing angle ($\psi = 45°$) in the small scheme. Within our framework this absence of exotics appears to be rather fortuitous.

4. Comparison with experiment

Consider now the Regge couplings. At present only a few of the results can directly be compared with experimental data.

Regge fits to π-N scattering yield [17] *

$$(C_f^{(+)})^{pp}_{\pi^+\pi^+} \approx -53.6 \pm 2\ . \tag{4.1}$$

The unknown fNN coupling is eliminated by using the result from N-N scattering [17]

$$(C_f^{(+)})^{pp}_{pp} \approx -120 \pm 2\ . \tag{4.2}$$

Under the assumption of factorization we obtain

$$C^{(+)}_{\pi\pi} \approx -24 \pm 3\ , \tag{4.3}$$

which agrees reasonably well with the value -32 predicted in our saturation scheme considering the roughness of the approximations involved. The corresponding determination of $C^{(-)}_{\pi\pi}$ is subject to more uncertainty. While π-N scattering gives a value of [17]

$$(C_\rho^{(-)})^{pp}_{\pi\pi} = -9.6 \pm (-0.6), \tag{4.4}$$

the determination of ρ- exchange in nucleons scattering is only bounded from above. One finds [17]

$$(C_\rho^{(-)})^{pp}_{pp} = 0^{+0}_{-23}\ . \tag{4.5}$$

Again employing factorization yields the upper bound

$$C^{(-)}_{\pi\pi} < -23\ .$$

* The small errors given in ref. [17] should obviously be taken with caution.

It is gratifying to note that at least this does not contradict our (exchange degenerate) prediction of ≈ -30. Recall that the Veneziano model [18] leads to the somewhat larger value [11]

$$C^{(+)}_{\pi\pi} = C^{(-)}_{\pi\pi} = -36 \ . \tag{4.6}$$

The coupling of the f-trajectory to $\rho\rho$ can be estimated by using the idea of vector-meson dominance together with the analysis of photoproduction of ρ^0 mesons.

For $\gamma p \to \rho^0 p$ the best fit [19] is

$$\left.\frac{d\sigma}{dt}\right|_{t=0} = A_P^2 \left[1 + \frac{a_f}{A_P}\,\frac{2}{\sqrt{E_\gamma/M}}\right] + O\left(\frac{a_f^2}{A_P^2}\right), \tag{4.7}$$

with

$$a_f/A_P \approx 0.8 \ ,$$

$$A_P^2 \approx 70\ \mu b/\mathrm{GeV}^2 \ .$$

From the vector-meson dominance relation

$$\left.\frac{d\sigma}{dt}(\gamma p \to \rho^0 p)\right|_{t=0} \approx \frac{\pi\alpha}{\gamma\rho^2}\left(\frac{1}{16\pi}\right)\sigma_T(\rho^0 p)\left[1 + \left(\frac{\mathrm{Re}\,T_{\rho p}}{\mathrm{Im}\,T_{\rho p}}\right)^2\right]$$

$$\approx \frac{1}{350}\left(\frac{1}{16\pi}\right)\sigma_T(\rho^0 p) \ , \tag{4.8}$$

$$\mathrm{Im}\,T^{\rho\rho}_{pp} = -4\sqrt{\pi}\,s\left(A_P + \frac{a_f}{\sqrt{E_\gamma/M}}\right) \approx -C^{(+)\rho\rho}_{pp}\left(\frac{\nu}{M^2}\right) \ , \tag{4.9}$$

we then have

$$\left(C_f^{(+)}\right)^{\rho\rho}_{pp} \approx -45 \ .$$

In order to obtain $\left(C_f^{(+)}\right)^{\pi\pi}_{\rho\rho}$ we multiply $\left(C_f^{(+)}\right)^{\rho\rho}_{pp}$ by

$$\left(C_f^{(+)}\right)^{\pi\pi}_{NN} \Big/ \left(C_f^{(+)}\right)^{NN}_{NN} \approx (-53.6)/(-120) \approx 0.45 \ ,$$

to obtain

$$C^{(+)\pi\pi}_{\rho\rho} \approx -20 \ . \tag{4.10}$$

Our calculations yields only *

$$C^{(+)\pi\pi}_{\rho\rho} \approx -10 \ . \tag{4.11}$$

The remaining Regge couplings should be tested in production experiments of higher resonances. For example the process $\pi p \to A_1 p$ can provide information about the $f(\pi A_1)$ and the $\rho(\pi A_1)$ couplings. Photo- and electroproduction of σ, f, A_1, A_2, B and D combined with the assumption of vector-meson dominance will give us information on many other vertices. In the absence of Regge analysis of such reactions we can compare here only with some results obtained from other models.

(i) The coupling of f and ρ in $\pi p \to A_1 p$ is predicted as follows: we have $C^{(+)}_{\pi A_1} \approx -12$ and $C^{(-)}_{\pi A_1} \approx -9$, and hence

$$(C^{(+)}_{f})^{pp}_{\pi^- A_1^-} = -12 \, \frac{C^{(+)NN}_{NN}}{C^{(+)\pi\pi}_{NN}} \approx -27 \ , \tag{4.12}$$

$$(C^{(+)}_{\rho})^{pp}_{\pi^- A_1^-} = -9 \, \frac{C^{(-)NN}_{NN}}{C^{(-)\pi\pi}_{NN}} \approx \begin{cases} 0 \\ -57 \end{cases} . \tag{4.13}$$

If we take the corresponding values from a $\pi\pi \to \pi A_1$ Veneziano model [20] we find the somewhat larger coupling

$$C^{(\pm)}_{\pi A_1} \approx -25.8 \ , \tag{4.14}$$

giving

$$(C^{(+)}_{f})^{pp}_{\pi^- A_1^-} \approx -57 \ , \tag{4.15}$$

$$(C^{(-)}_{\rho})^{pp}_{\pi^- A_1^-} \approx \begin{cases} 0 \\ -160 \end{cases} . \tag{4.16}$$

For a description of the model and a derivation of the coupling $C^{(\pm)}_{\pi A_1}$ see appendix B

(ii) The cross section for photoproduction of σ is predicted to be extremely small compared with photoproduction of ρ. The ρ- trajectory couples 16 times more weakly to $\gamma\sigma$ than to $\gamma\rho$ since from table 1:

$$C^{(-)}_{\rho\sigma} / C^{(-)}_{\rho\rho} = -\frac{1.2}{19.2} \approx -\frac{1}{16} \ .$$

Under appropriate smoothness assumptions this agrees with the result of broken

* Due to s-channel helicity conservation the experimental result gives the $\lambda = 1$ Regge coupling.

scale invariance [21] which tells us that σ cannot couple to a photon and a vector meson [22], and to two photons. That $\sigma\gamma\gamma$ is small is also found in dispersion relations with Compton amplitudes [23] which is related to $\sigma\gamma\rho$ by vector-meson dominance. Notice that in the photoproduction of f- mesons, ρ -exchange is predicted to be rather small. This statement will hopefully be tested soon.

5. Conclusion and outlook

Many phenomenological analyses will have to be performed in order to test the large number of predictions provided by our scheme. The few presently available estimates are compatible with the numbers presented in our tables.

It will be interesting to study the behaviour of the Regge couplings calculated with our method as the saturation scheme employed grows larger and larger. If the chiral wave functions are correct the rate of growth of $\{m_b^2, m_a^2\}$ and $[m_b^2, m_a^2]$ with N is controlled by the Regge intercepts $\alpha_f(0)$ and $\alpha_\rho(0)$. In a similar way, the exotic parts of the amplitudes will have to vanish rapidly as N increases. These requirements impose stringent conditions on possible extensions of the saturation scheme presented here.

In this work we have not considered any helicity-flip couplings of f-and ρ-mesons. They can be dealt with in a similar fashion. In this case the number of particles contained in our scheme is, however, inadequate even to allow for a suppression of the $I_t = 2$ exotics*.

A challenging question concerns the algebraic description of the pomeron, for which we have not found any answer as yet.

Let us finally mention that consideration of amplitudes such as $\pi\alpha \to \pi\pi\beta$ and $\pi\pi\alpha \to \pi\pi\beta$ will lead to the result that the algebra of all operators considered, viz., isospin T_a, axial charge X_a and the Regge couplings of f, ρ and A_1 trajectories, closes to one single larger Lie algebra. A sub-algebra of this may even coincide with the old conjecture of Cabibbo, Horwitz and Ne'eman [24] concerning an algebra** among f-and ρ-trajectories only.

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* The so called second Weinberg sum rule $[X, [X, mJ_+]]|_{I_t=2} = 0$, cannot be saturated with the particles of this saturation scheme for physical masses.

** Since the emergence of the concept of duality, Y. Ne'eman prefers to associate the scalar operator with the f-coupling rather than the pomeron. For the pomeron he now involves additional assumptions. See ref. [25].

Appendix A

Particle widths

The decay width of a particle $\gamma \to \alpha + \pi$ is given from PCAC in terms of our coupling matrix $G^{I_\gamma I_\alpha}_{\gamma\alpha}$ by

$$\Gamma_{\gamma\to\alpha\pi} = f_l R \; \frac{(m_\gamma^2 - m_\alpha^2)^3}{m_\rho^3 \; m_\gamma^3} \; \frac{1}{(2J_\gamma + 1)} \sum_\gamma \begin{Bmatrix} |G^{10}_{\gamma\alpha}(\lambda)|^2 \\ 3|G^{01}_{\gamma\alpha}(\lambda)|^2 \\ 2|G^{11}_{\gamma\alpha}(\lambda)|^2 \end{Bmatrix} , \quad \text{for} \begin{cases} 1 \to 0 \\ 0 \to 1 \\ 1 \to 1 \end{cases} , \qquad \text{(A.1)}$$

where $R = m_\rho^3/16\pi f_\pi^2 \approx 1$ MeV and f_l is a threshold factor*

$$f_l = \left[1 - \frac{2m_\pi^2 (m_\gamma^2 + m_\alpha^2)}{(m_\gamma^2 - m_\alpha^2)^2}\right]^{l+\frac{1}{2}} \qquad \text{(A.2)}$$

correcting for the finite pion mass in an orbital angular momentum l.

Inserting our solutions for G we find (using $m_\omega^2 = m_\rho^2, m_{A_1}^2 \approx 2m_\rho^2, m_{A_3}^2 \approx 2.9\, m_\rho^2$, $m_f^2 \approx 2.7\, m_\rho^2, m_B^2 \approx 2.6\, m_\rho^2$):**

$$\Gamma_{\rho\pi\pi} = \left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{\frac{3}{2}} \tfrac{1}{3} R\, |G_{\rho\pi}(0)|^2 \approx 0.8 \times 330 \cos^2\psi \approx 265 \cos^2\psi \text{ MeV}, \qquad [135 \pm 20] ,$$

$$\Gamma_{\sigma\pi\pi} = \left(1 - \frac{4m_\pi^2}{m_\sigma^2}\right)^{\frac{1}{2}} \left(\frac{3R}{2}\right)\left(\frac{m_\sigma}{m_\rho}\right)^3 |G_{\sigma\pi}(0)|^2 = \left(1 - \frac{4m_\pi^2}{m_\sigma^2}\right)^{\frac{1}{2}}$$

$$\times\, 1500 \left(\frac{m_\sigma}{m_\rho}\right)^3 \sin^2\psi \text{ MeV} , \qquad [?] ,$$

$$\Gamma_{A_1\rho\pi} = f^{(2)} \left(\frac{R}{3\sqrt{2}}\right)(|G_{\rho A_1}(0)|^2 + 2|G_{\rho A_1}(1)|^2) \approx 0.575 \times 236(\sin^2\phi + \sin^2\psi) \text{ MeV}$$

$$= 135\,(\sin^2\phi + \sin^2\psi) \text{ MeV}, \qquad [80 \pm 30] ,$$

$$\Gamma_{A_1\sigma\pi} = f^{(1)}\left(\frac{R}{6\sqrt{2}}\right)\left(2 - \frac{m_\sigma^2}{m_\rho^2}\right)^3 |G_{\sigma A_1}(0)|^2 \approx f^{(1)} \times 80\left(2 - \frac{m_\sigma^2}{m_\rho^2}\right)^3 \cos^2\psi \text{ MeV}, \quad [?] , \qquad \text{(A.3)}$$

* In addition, if $\alpha \equiv \pi$, there is a Bose factor $\frac{1}{2}$ and f_l becomes $(1 - 4m_\pi^2/m_\gamma^2)^{l+\frac{1}{2}}$. In the numerical evaluation we have displayed the threshold factors separately to show their effect on the decay widths.

** In evaluating the $A_1 \to \rho\pi$ width we have included only a d-wave threshold factor since the A_1 decays predominantly in the d-wave. The s- and d-wave coupling constants are linear combinations of the above helicity coupling constants.

$$\Gamma_{B\omega\pi} = f^{(1)}\left(\frac{R}{3}\right)(|G_{\omega B}(0)|^2 + 2|G_{\omega B}(1)|^2) \approx 0.86 \times 660 \sin^2\phi$$

$$= 570 \sin^2\phi \text{ MeV}, \qquad [100 \pm 20],$$

$$\Gamma_{f\pi\pi} = f^{(2)} R \times 1.32 |G_{\pi f}(0)|^2 \approx 0.88 \times 440 \sin^2\psi \approx 390 \sin^2\psi \text{ MeV}, \qquad [125 \pm 20],$$

$$\Gamma_{A_2\rho\pi} = f^{(2)} R \times 0.56 (|G_{\rho A_1}(0)|^2 + 2|G_{\rho A_2}(1)|^2) \approx 0.83 \times 560 \sin^2\phi$$

$$= 463 \sin^2\phi \text{ MeV}, \qquad [76.8 \pm 15],$$

$$\Gamma_{\omega\to 3\pi} = 4 \times 10^{-8} \text{ MeV} \times g^2_{\rho\omega\pi}, \qquad [10.3 \pm 0.8],$$

where

$$g_{\rho\omega\pi} \equiv \frac{2}{f_\pi} \cos\phi .$$

Here $g_{\omega\rho\pi}$ is the coupling constant introduced by Gell-Mann, Sharp and Wagner [26] *. In the above experimental decay widths have been added in the square brackets. For $A_1 \to \rho\pi$ decay it is not clear how much the decay mode $A_1 \to \sigma\pi$ contributes to the total width. We see that choosing $|\sin\psi| = 1/\sqrt{3}$, $|\sin\phi| = 1/\sqrt{6}$ gives good agreement with all known widths.

In addition, if we assume the σ to be $\approx$ 700 MeV we predict a width of about 500 MeV in agreement with most phenomenological analyses of $\pi\pi$ scattering. It is the same meson that has been the subject of much speculation in connection with broken scale invariance [21]. In the literature there also exist models that lead to a low-lying σ-meson of mass $m_\sigma \approx 400$ MeV (ref. [27]). From our formula for $\Gamma_{\sigma\pi\pi}$ we see that this σ-meson would have a width of only about 100 MeV. We consider it improbable that such a narrow resonance would have escaped detection until now. Notice that this small width predicted for a low-lying σ-meson is by no means a consequence of our specific saturation model. It follows quite generally from the Adler-Weisberger relation for π-π scattering in the narrow resonance approximation

$$G^2_{\sigma\pi} + G^2_{\rho\pi} + \dots\dots = 1 , \qquad \text{(A.4)}$$

with only positive contributions. The experimental $\rho\pi\pi$ width gives $G^2_{\rho\pi} = (g_{\rho\pi\pi} f_\pi/m_\rho)^2 \approx \frac{1}{2}$, leading to

* Their effective Lagrangian is

$$L_{\omega\rho\pi} = g_{\omega\rho\pi} \epsilon_{\mu\nu\lambda\delta} \partial^\mu \omega^\nu \partial^\lambda \rho^\delta \pi , \qquad \epsilon_{0123} = 1 .$$

Hence the connection of $g_{\omega\rho\pi}$ with $G_{\omega\rho}(1)$ is $g_{\omega\rho\pi} = (2/f_\pi) G_{\omega\rho}(1)$.

$$\frac{\Gamma_{\sigma\pi\pi}}{m_\sigma} < \left(\frac{m_\sigma}{m_\rho}\right)^2 \left(1 - \frac{4m_\pi^2}{m_\sigma^2}\right)^{\frac{1}{2}} , \tag{A.5}$$

thereby forcing any light σ-meson to be very narrow.

Further, we wish to quote the experimental information* on the ratio of the longitudinal to the transverse coupling in the $A_1 \to \rho\pi$ decay [28]:

$$\left| \frac{G_{A_1\rho}(1)}{G_{A_1\rho}(0)} \right| = 0.48 \pm 0.13 \ . \tag{A.6}$$

In this theory this ratio is given by $\sin\phi/(\sqrt{2}\sin\psi) = \frac{1}{2}$. Notice that in the specific mixing model of ref. [4] exactly the same ratio is predicted. There is yet another experimental determination of $|G(1)/G(0)|$ by Crennel et al. [30]. These authors find

$$\left| \frac{G_{A_1\rho}(1)}{G_{A_1\rho}(0)} \right| = 0.87 \begin{matrix} +0.07 \\ -0.06 \end{matrix} , \tag{A.7}$$

with a positive relative phase of $G(1)$ and $G(0)$ preferred. Such a large ratio would yield too large an angle ϕ to be compatible with the $B \to \omega\pi$ decay width [see eq. (A.3)]. We shall therefore adhere to the value (A.6) with a positive relative phase. The connection of $G(0)$ and $G(1)$ with the standard couplings $g_{A_1\rho\pi}$ and $h_{A_1\rho\pi}$ is given by**

$$\begin{aligned} G(0) &= -\kappa\sqrt{2}g_L = \kappa\frac{\sqrt{2}}{m_\rho}(-h + 6g) , \\ G(1) &= +\kappa g_T = \kappa\frac{8g}{m_\rho} , \end{aligned} \tag{A.8}$$

where

$$\kappa = \tfrac{1}{8} f_\pi \ .$$

With the angles ψ and ϕ determined above we find

$$\begin{aligned} g &\approx -\ 2.3 , \\ h &\approx \ 12.3 . \end{aligned} \tag{A.9}$$

Finally we wish to point out that this mixing scheme predicts the B-meson to decay into $\omega\pi$ only in the transverse mode. This is in agreement with the experiment of Ascoli et al. [31]. Their most recent determination yields

* One also finds the statement [29] that ref. [28] misquoted their result. It actually should read $|G_{A_1\rho}(1)/G_{A_1\rho}(0)| = 0.565 \pm 0.35$.

** See eq. (B.4) for the definitions. In terms of g and h the width $\Gamma_{A_1\rho\pi} = 6.5[g^2 - 0.12\,gh + 0.01\,h^2]$ MeV.

$$\frac{\Gamma_L(B\to\omega\pi)}{\Gamma_T(B\to\omega\pi)} \approx 0.189 \pm 0.052 \ . \tag{A.10}$$

Appendix B

Veneziano model for $\pi\pi\to\pi A_1$

Here we calculate the residues $C^{(\pm)}_{\pi A_1}$ given by a standard Veneziano model for $\pi\pi\to\pi A_1$ scattering [20]. We describe the process

$$\pi_a(p_1)+\pi_b(p_2)+\pi_c(p_3)\to A_{1\alpha}(k,\lambda)\ ,$$

by

$$T=\epsilon_\mu(k,\lambda)\,T^\mu_{abc,d}\ ,$$

where T_μ is the completely crossing symmetric amplitude

$$T_\mu = I_{abcd}\,[(p_2-p_3)_\mu\,\alpha(s,t)-p_{1\mu}\,\beta(s,t)]$$
$$+\,(\text{cyclic terms } 1\to2\to3,\ s\to t\to u,\ a\to b\to c)\ , \tag{B.1}$$

with

$$I_{abcd}=\tfrac{1}{2}[\delta_{ab}\,\delta_{cd}+\delta_{ac}\,\delta_{bd}-\delta_{ad}\,\delta_{bc}]\ , \tag{B.2}$$

being the isospin projection operators free of $I_s=2$ and $I_t=2$ contributions*, and $\alpha(s,t)$ and $\beta(s,t)$ are crossing anti-symmetric and symmetric functions, respectively. The minimal Veneziano ansatz consistent with factorization in the leading Regge trajectories of the processes $\pi\pi\to\pi\pi$, $\pi\pi\to\pi A_1$ and $\pi A_1\to\pi A_1$, and containing the correct Adler zeros and current algebra constraints is given by**

$$\alpha(s,t)=\tfrac{1}{2}(g(s,t)-g'(s,t))\ ,\qquad \beta(s,t)=-\tfrac{1}{2}(3g(s,t)+g'(s,t))\ , \tag{B.3}$$

where $g(s,t)$ and $g'(s,t)$ are determined completely in terms of $\rho\pi\pi$ and $A_1\rho\pi$ coupling constant with the standard definition

$$L=g_{\rho\pi\pi}\,\boldsymbol{\rho}^\mu\,\boldsymbol{\pi}\times\partial_\mu\,\boldsymbol{\pi}+g_{A_1\rho\pi}\,m_\rho\,\boldsymbol{\rho}_\mu\,A^\mu\times\boldsymbol{\pi}+h_{A_1\rho\pi}\,\frac{1}{m_\rho}\,\boldsymbol{\rho}_\mu\,\partial^\mu A^\nu\times\partial_\nu\boldsymbol{\pi}\ . \tag{B.4}$$

One finds two possible amplitudes given by

$$g(s,t)\left(\frac{m_\rho}{g_{\rho\pi\pi}}\right)=\left[(-\tfrac{1}{3}g\pm\tfrac{1}{6}h)\left(\frac{s}{2m_\rho^2}\right)-\frac{h}{2}\left(\frac{t}{2m_\rho^2}\right)\right]B_{12}(s,t)+g\left(\frac{t}{2m_\rho^2}\right)B_{21}(t,u)$$
$$+\left[(-\tfrac{2}{3}g\pm\tfrac{1}{3}h)\left(\frac{s}{2m_\rho^2}\right)+(2g+h)\left(\frac{t}{2m_\rho^2}\right)\right]B_{22}(s,t)\ , \tag{B.5}$$

* Hence $T^{I_u=2}_\mu=(p_2-p_3)_\mu\,\alpha(s,t)-p_{1\mu}\,\beta(s,t)$.
** See the last of ref. [20].

$$g'(s,t)\left(\frac{m_\rho}{g_{\rho\pi\pi}}\right) = \left[(\tfrac{7}{3}g \mp \tfrac{1}{6}h)\left(\frac{s}{2m_\rho^2}\right) + \frac{h}{2}\left(\frac{t}{2m_\rho^2}\right)\right] B_{11}(s,t)$$

$$+ \left[-h\left(\frac{s}{2m_\rho^2}\right) + (-\tfrac{5}{3}g \pm \tfrac{1}{3}h)\left(\frac{t}{2m_\rho^2}\right)\right] B_{21}(s,t)$$

$$+ \left[(\tfrac{14}{3}g + 2h \mp \tfrac{1}{3}h)\left(\frac{s}{2m_\rho^2}\right) + (-\tfrac{10}{3}g \pm \tfrac{2}{3}h - h)\left(\frac{t}{2m_\rho^2}\right)\right] B_{22}(s,t) \ . \tag{B.6}$$

Here $B_{mn}(s,t)$ is the usual beta-function

$$B_{mn}(s,t) = \frac{\Gamma(m-\alpha(s))\,\Gamma(n-\alpha(t))}{\Gamma(m+n-\alpha(s)-\alpha(t))} \ . \tag{B.7}$$

Let us now calculate the s-channel helicity amplitude. We find for $I_t = 0$ and $I_t = 1$

$$\epsilon^\mu(0)\, T_\mu^{I_t=0} = \left(\frac{s-u}{2M_A}\right)\left[-\tfrac{1}{2}\alpha(s,u) - \tfrac{3}{4}(\alpha(t,s)-\alpha(t,u)) - \tfrac{3}{4}(\beta(t,s)-\beta(t,u))\right]$$

$$+ \tfrac{1}{2}M_A\left[-\tfrac{1}{2}\beta(s,u) + \tfrac{9}{4}(\alpha(t,s)+\alpha(t.u)) - \tfrac{3}{4}(\beta(t,s)+\beta(t,u))\right] \ , \tag{B.8}$$

$$\epsilon^\mu(0)\, T^{I_t=1} = \left(\frac{s-u}{2M_A}\right)\left[-\tfrac{1}{2}(\alpha(t,s)+\alpha(t,u)) - \tfrac{1}{2}(\beta(t,s)+\beta(t,u))\right]$$

$$+ \tfrac{1}{2}M_A\left[\tfrac{3}{2}(\alpha(t,s)-\alpha(t,u)) - \tfrac{1}{2}(\beta(t,s)-\beta(t,u))\right] \ . \tag{B.9}$$

As a consequence the asymptotic behaviour of the imaginary parts for $s \to \infty$ are given by eq. (2.12) with

$$C_{\pi A_1}^{(\pm)} = -\frac{\sqrt{\pi}\, g_{\rho\pi\pi}\, m_\rho}{4M_A}\left[-2g_{A_1\rho\pi} \pm h_{A_1\rho\pi} + 2h\right] \ . \tag{B.10}$$

Using the experimentally determined ratio $|G_{\rho A_1}^{(1)}/G_{\rho A_1}^{(0)}(0)| \approx 0.48$ or $g/h \approx -\frac{1}{5.77}$ we finally obtain

$$C_{\pi A_1}^{(\pm)} \approx -\begin{cases} 25.8 \\ 35.0 \end{cases} , \qquad \text{for} \qquad \Gamma_{A_1\rho\pi} = \begin{cases} 80 \\ 140 \end{cases} \text{MeV} \tag{B.11}$$

for the lower sign, and

$$C_{\pi A_1}^{(\pm)} \approx \begin{cases} -63.4 \\ -84.0 \end{cases} \tag{B.12}$$

for the upper sign in eqs. (B.5) and (B.6). Comparing with our predictions

$$C^{(+)}_{\pi A_1} \approx -12.0\,, \qquad C^{(-)}_{\pi A_1} \approx -9.0\,, \tag{B.13}$$

we see that the lower sign at a width of 80 MeV is to be preferred. For completeness we note that with the values $g = -2.3$ and $h = +12.3$ predicted by our scheme [see eq. (A.9)] the Veneziano model yields:

$$C^{(\pm)}_{\pi A_1} \approx \begin{cases} -24.1 \\ \\ -59.1 \end{cases} \tag{B.14}$$

for the lower and upper signs respectively.

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