THE TWO SUPERFLOWS IN ^{3}He-A

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The topological aspects of superflow are discussed. While the barrier of condensation energy stabilizes only flux zero and flux one, the coupling of current and l-texture leads, in a torus and for the limited range of temperature and magnetic field in which a helical texture exists, to the separate quasi-topological stabilization of *both* the macroscopic pair current and the bound orbital current $\nabla \times l$. The latter decays much faster than the first and has a much smaller critical value. The decay rate is calculated.

The order parameter of ^{3}He-A, $d_a(\phi^{(1)} + i\phi^{(2)})_i$, is specified by the position of the dreibein $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}$, $\boldsymbol{l} = \boldsymbol{\phi}^{(1)} \times \boldsymbol{\phi}^{(2)}$ and the direction of the unit vector $\boldsymbol{d}$, except for a simultaneous reflection of $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \boldsymbol{d}$ (which preserves $\boldsymbol{l}$). Physically, $\boldsymbol{l}$ describes the direction of orbital angular momentum of the Cooper pairs and $\boldsymbol{d}$ the axis along which their total spin has zero component. The parameter space is topologically equivalent to $SO_3 \times S^2/Z_2$, where SO_3 may be pictured in the axis-angle form as a sphere of radius π with diametrically opposite points identified (i.e. the projective space P_3). Every vector $\boldsymbol{a}$, $|\boldsymbol{a}| \leqslant \pi$ fixes a rotation $e^{-i\boldsymbol{a}\boldsymbol{L}}$ by which the dreibein differs from a standard position, say that coinciding with the x, y, and z axes. The surface of a sphere S^2 is the locus of the end points of all possible $\boldsymbol{d}$ vectors. The center Z_2 consists in the reflection defined above.

It was noted some time ago [1] that this parameter space allows only for four types of inequivalent closed contours (i.e. $\pi_1 = Z_4$) such that the superliquid ^{3}He-A in a torus could only support four different topologically stable fluxes $0, \pm\frac{1}{2}$, and 1, which are all of microscopic size. Thus it did not even seem to deserve the prefix "super" in its name. Only an infinite connectedness (i.e. $\pi_1 = Z$) is capable of topologically stabilizing arbitrarily high quantum numbers, $N = 1,2,3, \ldots$, which can accumulate to a macroscopic superflow. The situation becomes even worse by observing that in a bulk liquid the dipole force aligns the $\boldsymbol{d}$ and $\boldsymbol{l}$ vectors thus reducing the parameter space to SO_3 only. Now there are only two classes of topologically inequivalent closed contours (i.e. $\pi_1 = Z_2$) as is well known from the discussion of half-integer spin representations.

Certainly, topological stability is not absolute. It is derived on the assumption of an infinite condensation energy. In fact, topological quantum numbers *do* decay by penetration through some barrier in functional space. This barrier is gigantic due to the presence of volume factors. It was soon realized that such volume factors can appear in potential barriers much smaller than the condensation energy thereby causing a very long lifetime of metastable states. A uniform current in ^{3}He-A is indeed stabilized by such a mechanism. For, the free energy [2]

$$
\begin{aligned}
f = {} & \tfrac{1}{2}\rho_s \boldsymbol{v}_s^2 - \tfrac{1}{2}\rho_s^0 (\boldsymbol{l}\cdot\boldsymbol{v}_s)^2 \\
& + c\boldsymbol{v}_s\cdot(\nabla\times\boldsymbol{l}) - c^0(\boldsymbol{v}_s\cdot\boldsymbol{l})\boldsymbol{l}\cdot(\nabla\times\boldsymbol{l}) \\
& + \tfrac{1}{2}\rho_{sp}(\partial d_a)^2 - \tfrac{1}{2}\rho_{sp}(\boldsymbol{l}\partial d_a)^2 \\
& + \tfrac{1}{2}K_s(\nabla\cdot\boldsymbol{l})^2 + \tfrac{1}{2}K_t[\boldsymbol{l}\cdot(\nabla\times\boldsymbol{l})]^2 \\
& + \tfrac{1}{2}K_b[\boldsymbol{l}\times(\nabla\times\boldsymbol{l})]^2 - g_d(\boldsymbol{l}\cdot\boldsymbol{d})^2 + g_z(H\boldsymbol{d})^2, \qquad (1)
\end{aligned}
$$

contains a term $-\frac{1}{2}\rho^0(\boldsymbol{l}\cdot\boldsymbol{v}_s)^2$ which represents a barrier to the motion of $\boldsymbol{l}$ orthogonal to $\boldsymbol{v}_s$. This may be pictured as a potential hill around the equator of the

SO_3 parameter sphere. But without this motion there can be no relaxation of flow: the deformation of all contours to the four (or two) inequivalent ones requires precisely the passage of the equator. This barrier is quite insurmountable due to volume factors. If it is *idealized* to be infinitely high, just as previously the large but finite condensation energy, the stability problem can again be answered by topological arguments. In the case at hand, a *complete* elimination of the equator renders the SO_3 sphere infinitely connected ($\pi_1 = Z$) permitting the build-up of a macroscopic number of flux quanta.

Recently [3,4] it was pointed out that the $\boldsymbol{l} \parallel \boldsymbol{v}_s$ alignment force has to compete with the other terms in eq. (1) which may destroy the previous argument. In fact, the stability holds at zero magnetic field if a combination of hydrodynamic parameters satisfies

$$K \equiv \rho_0 K_b/(c_0 + \tfrac{1}{2}\rho_s^{\parallel})^2 > 1\,, \qquad (2)$$

which is true close to T_c (where $K = \frac{10}{9}$). For $T < T_{stab} \approx 0.86\, T_c$, however, the alignment is lost.

Fortunately, it can be shown that $\boldsymbol{l}$ does not move away arbitrarily from the current direction but stabilizes at a new helical texture [5] with $\boldsymbol{l}$ winding around the average current $j \equiv \langle J \rangle$ of

$$J = (\hat{\rho}_s \boldsymbol{v}_s + \mathbf{c}(\nabla \times \boldsymbol{l}))/J_d\,, \qquad (3)$$

measured in units of $J_d = \rho_s^{\parallel} v_d$ with $v_d \equiv 1/2m\xi_d$ being the dipole velocity (≈ 0.1 cm/s).

In the presence of the magnetic field $H = hH_d$ (with $H_d \approx 30$ Oe being the field at which the magnetic energy equals the dipole alignment energy) the criterion for the onset of a helical texture is (see fig. 1) [6,7]

$$\frac{h^2}{j^2} > \frac{h_c^2}{j^2} \equiv \frac{1}{2}\frac{\rho_0}{\rho_s^{\parallel}}(1 - K^{-1})\,. \qquad (4)$$

If this condition is fulfilled the helix forms at an angle of inclination [‡1]

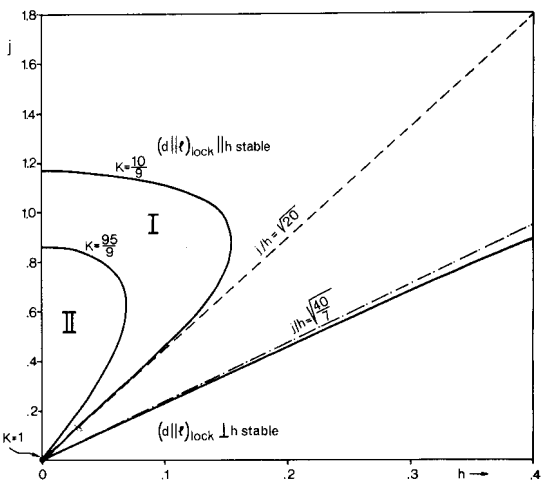

Fig. 1. Regions of stable superflow j in the presence of a magnetic field $\boldsymbol{h} \parallel \boldsymbol{j}$. The region $h^2/j^2 < 1/20$ is stable for complete dipole locking with $\boldsymbol{d} \parallel \boldsymbol{l} \parallel \boldsymbol{h} \parallel \boldsymbol{j}$ at $T \approx T_c$. In region I dipole locking is relaxed ($T \approx T_c$). For $T_i \in (T_{stab}, T_c)$ with $\kappa = 95/90$ this region shrinks to II. For $T = T_{stab}$ the alignment collapses. The lower triangle to the right shows the stability of the $\boldsymbol{d} \parallel \boldsymbol{l} \perp \boldsymbol{h} \parallel \boldsymbol{j}$ texture for $T \approx T_c$. If dipole locking is relaxed the boundary turns smoothly into the line $j = j_{max} \approx (40/7)^{1/2} \approx 2.39$.

$$\beta_h^2 \approx \frac{2}{D}\frac{\delta_h}{j^2} \approx 3\,\frac{h^2 - h_c^2}{j^2} \approx \left(1 - \frac{T}{T_{stab}}\right) + 3\frac{h^2}{j^2}\,. \qquad (5)$$

where

$$D = \rho_0[(c_0 + \tfrac{1}{2}\rho_s^{\parallel})(2K_t - \tfrac{1}{2}\rho_s^{\parallel}) - \tfrac{3}{2}\rho_s^{\parallel} K_b]/\rho_s^{\parallel} K_b (c_0 + \tfrac{1}{2}\rho_s^{\parallel})\,,$$

and $\delta_h = h^2 - h_c^2$.

For stability it is necessary that the pitch $\partial\gamma/\partial z$ of the helix (γ = azimuthal angle of $\boldsymbol{l}$) varies in some limited range (see fig. 2) close to:

$$\partial\gamma(z)/\partial z \approx rj\,, \qquad (6)$$

with $r \equiv (c_0 + \frac{1}{2}\rho_s^{\parallel})/K_b$, if also z is measured in dipole units ξ_d ($\approx 10^{-3}$ cm).

The whole stability argument is valid only as long as the current stays sufficiently below the dipole value [8] since for $j > 1$ the helix curls up so much that the bending energies, which prefer a flat $\boldsymbol{d}$ texture, tear $\boldsymbol{d}$ away from $\boldsymbol{l}$, dipole locking is lost and the stability region disappears (see fig. 1).

It is the purpose of this note to discuss the implica-

[‡1] I thank Prof. K. Maki for pointing out that R. Kleinberg at La Jolla has apparently measured $\beta_h \neq 0$ by sending a sound signal transverse to the current. He observes the attenuation constants [9]

$$\alpha = \alpha + (\alpha_c - \alpha_\perp)\sin^2\beta_h + (\alpha_\parallel - 2\alpha_c + \alpha_\perp)\cdot\tfrac{3}{8}\sin^4\beta_h\,,$$

with a slope in $h^2 - h_c^2$ compatible with eq. (5).

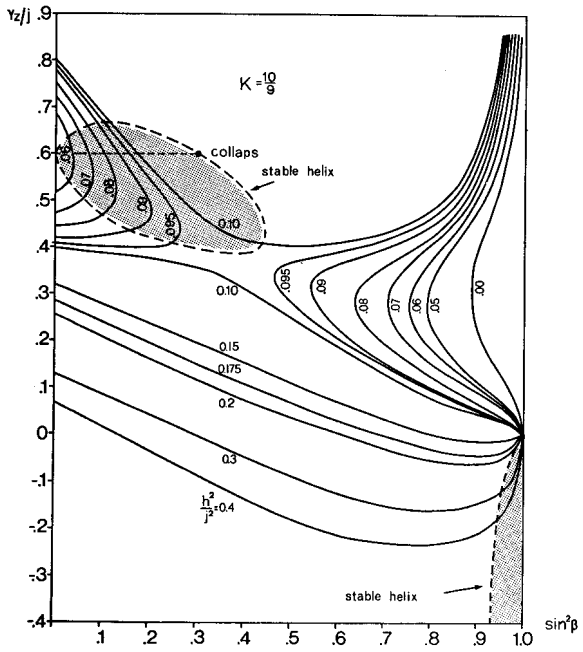

Fig. 2. Lines of stationary solutions γ_z/j and $\sin^2\beta_h$ for $T \approx T_c$ and different values of h^2/j^2 in the dipole locked regime. The shaded areas correspond to stable helical textures. As h^2 becomes larger than h_c^2, the helix may form in the neighbourhood of $\gamma_z/j = 3/5$. If h is increased faster than Γ (see eq. (20)), the angle between l and j opens up until about $\sin^2\beta_h \approx 0.32$ where the helix collapses. If the temperature is lower than T_c, the small-s region looks almost the same, except that the numbers on the curves correspond to $(h^2 - h_c^2)/j^2 + 1/20$ instead of h^2/j^2.

tions of the helix to the superflow properties of ^{3}He.

Let us first observe that the current (3) has two pieces analogous to the source of a magnetic field in the presence of magnetizable matter: the first piece describes the flow of Cooper pairs (pair current), the second gives the apparent particle transport resulting from the orbital circulation of the atoms *within* each pair (orbital current). If we assume, for simplicity, a purely z-dependent texture, the total current runs in the z-direction while the curling-up of the helix gives the apparent particle flux as **c** times

$$\nabla \times \boldsymbol{l} = l^{\perp}\gamma_z \,. \tag{7}$$

The point is now that the helix makes also the second part of the current *topologically* stable with an energy barrier even smaller than the previous one (which prevented the decay of j) but still large due to volume factors. This is easily seen: the helix forms since the free energy for $h^2 > h_c^2$ has a potential minimum at $\beta = \beta_h$ and a potential mountain for $\beta = 0$. This barrier presents an obstacle to the l vector pointing in forward direction. With respect to the SO_3 parameter space, this amounts to a narrow cigar-shaped region being forbidden within the SO_3 parameter space (in addition to the equatorial line). Idealizing this cigar to be impenetrable, there is obviously a second infinite connectedness: paths winding an arbitrary number of times around the cigar are inequivalent ($\pi_1 = Z + Z$). Thus, we conclude that in the helical texture ^{3}He-A is a *double*-superliquid.

Certainly, since the barrier has finite height this "topological" conservation law is broken by tunneling amplitudes. Let us briefly discuss the decay properties. For simplicity, we shall assume j to be much smaller than unity such that the dipole force causes complete alignment of $\boldsymbol{d}$ and $\boldsymbol{l}$. Then the parameter space consists only of the dreibein $\boldsymbol{\phi}^{(1)}, \boldsymbol{\phi}^{(2)}, \boldsymbol{l}$ or the Euler angles β, γ and α. The particle current for small β can be written as

$$J \approx \alpha_z + \gamma_z \,, \tag{8}$$

and the free energy density, measured in units of $f_d \equiv \rho_s^{\parallel} v_d^2$, becomes for small β

$$2(f - j\alpha_z) \approx -j^2 + 2j\gamma_z + (K_b/\rho_s^{\parallel})\beta_z^2 + (K_b/\rho_s^{\parallel})\beta^2(\gamma_z - rj)^2 - 2\delta_h(\beta^2 - \beta^4/2\beta_h^2)\,, \tag{9}$$

where $r = (c_0 + \frac{1}{2}\rho_s^{\parallel})/K_b$. Now, this energy contains a second divergenceless current

$$J_2 = (K_b/\rho_s^{\parallel})\beta^2(\gamma_z - rj)\,, \tag{10}$$

which leads to a free energy density

$$2(f - j\alpha_z) = -j^2 + 2j\gamma_z + 4g_h(2\tilde{g})\,, \tag{11}$$

with

$$2\tilde{g} = \phi_x^2 - \phi^2 + \tfrac{1}{2}\phi^4 + \tilde{J}_2^2/\phi^2 \,. \tag{12}$$

The factor g_h is the density of "condensation energy" (per ξ_d^3) for the second order phase transition to the helical texture:

$$g_h = \frac{1}{D}\frac{\delta_h^2}{j^2} = \frac{1}{4}\frac{K_b}{\rho_s^{\parallel}}\frac{\beta_h^2}{\xi_h^2}. \tag{13}$$

The coordinate x is measured in units of the corresponding coherence length,

$$\xi_h = [K_b/2\rho_s^{\parallel}\delta_h]^{1/2}, \tag{14}$$

and $\tilde{J}_2 \equiv (4g_h \cdot \xi_h)J_2$, i.e.

$$\tilde{J}_2 = \phi^2(\gamma_x - rj\xi_h). \tag{15}$$

Let $\tilde{J}_2 \equiv \kappa(1-\kappa^2)$. Then, for $\kappa^2 < \frac{1}{3}$ there is an extremal solution

$$\phi_0(x) \equiv (1-\kappa^2)^{1/2}, \quad \gamma_0(x) = rj\xi_h x + \kappa x, \tag{16}$$

with

$$2\tilde{g}_0 = -(1-\kappa^2)^2/2, \tag{17}$$

which extremizes the free energy. This solution describes the metastable helical texture with a superflow J_2 (in addition to the total superflow J). This second superflow can decay only by β tunneling into the small β region. Such an excursion is extremal for the solution of the equations of motion [10]:

$$\phi_b^2(x) = 1 - \kappa^2 - (1-3\kappa^2)/\mathrm{ch}^2((1-3\kappa^2)/2)^{1/2}x,$$

$$\gamma_b(x) = rj\xi_h x + \kappa x + \mathrm{arctg}[((1-3\kappa^2)/2\kappa^2)^{1/2}\,\mathrm{th}((1-3\kappa^2)/2)^{1/2}x]. \tag{18}$$

This solution describes the statistical formation of a critical bubble of radius $r_c \approx \xi_h\xi_d/(1-3\kappa^2)^{1/2}$ whose outside is superliquid with respect to the current J_2, with the inside being almost normal. If the system is contained in a torus of length l (measured in units $\xi_h\xi_d$) and periodic boundary conditions are applied to γ ($\gamma(l/2) = \gamma(-l/2) + 2\pi n$), the bubble-free solution (16) must be quantized with wave number $\kappa_n = 2\pi n/l - rj\xi_h$. The bubbles, on the other hand, can decrease or increase the current to $\tilde{J}_2 = \kappa_n^{<,>}(1-(\kappa_n^{<,>})^2)$ depending on whether

$$\kappa_n^{<,>} = \kappa_n - \frac{2}{l}\left(\mathrm{arctg}\left(\frac{1-3\kappa_n^2}{2\kappa_n^2}\right)^{1/2} - \begin{Bmatrix} 0 \\ \pi \end{Bmatrix}\right)$$

$$\equiv \kappa_n - \frac{2}{l}\left(\delta_0 - \begin{Bmatrix} 0 \\ \pi \end{Bmatrix}\right), \tag{19}$$

where 0 corresponds to $\kappa^<$ and π to $\kappa^>$. Due to the piece $j\gamma_z$ in the free energy, the current increasing transition is greatly suppressed and the decay rate is found to be

$$\Gamma \approx \frac{l}{\pi\tau}\sqrt{\frac{3}{2}}\left(\frac{F_b}{2\pi T}\right)^{1/2} \times \exp[-(F_b - 8F_h\kappa(1-\kappa^2)\delta_0)/T], \tag{20}$$

where $1/\tau = \mu$ is the orbital viscosity, F_b is the bubble energy,

$$F_b = F_h \cdot \tfrac{8}{3}\sqrt{2}(1-3\kappa^2)^{1/2}, \tag{21}$$

and F_h is the total condensation energy in the volume $(\sigma\xi_d)\xi_h$ (i.e. $F_h = (\sigma\xi_d)f_d g_h \xi_h$). Starting at some current within the domain of stability $\kappa^2 < \frac{1}{3}$, the decay proceeds until κ hits the lower boundary of this domain ($\kappa = -1/\sqrt{3}$) (see figs. 2, 3).

Since the coherence length can be made much larger than ξ_d by going with h close to h_c, tubes of diameter 0.1 mm may well act as coherent one-dimensional objects similar to thin superconducting wires.

The presence of bubble solutions mediating the decay of superflow may be observable in changes of the attenuation of sound along the helix.

Notice that the superflow j would decay by a similar mechanism even though with a much longer lifetime: penetration is needed through the much larger barrier $\beta \approx \pi/2$ in order to relax one unit of flow.

Similar considerations hold for strong magnetic fields. Here we only mention that the $\boldsymbol{d} \parallel \boldsymbol{l} \perp \boldsymbol{h} \parallel \boldsymbol{j}$ position is stable for

$$j^2 \leqslant 2\frac{\rho_s^2}{\rho_s^{\parallel}}\frac{h^2}{h^2+1}\Big/\left(\rho_0 + \frac{(\rho_s - c_0)^2}{K_t}\right) \approx \tfrac{40}{7}\,h^2/(h^2+1), \tag{22}$$

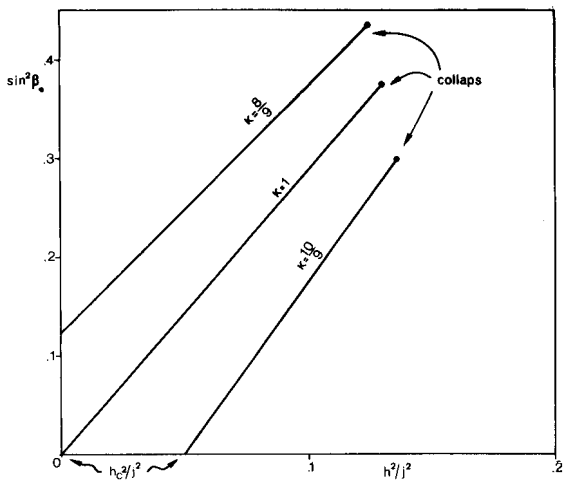

Fig. 3. The dependence of the angle between l and j as a function of h^2/j^2 for different values of temperature $T \approx T_c$, $T \approx T_{stab}$, $T < T_{stab}$ with $K = 10/9, 1, 8/9$, respectively, if γ_z starts out at $\gamma_z \approx (3/5)j$ and the changes of h proceed much faster than the decay rate of the helix.

where the initial linear piece corresponds to the dipole locked limit (see the lower region to the right of fig. 1). Moreover, as we see in fig. 2, there is a whole neighbourhood of $\beta_0 \approx \pi/2$ which is stable as well, namely all helices with pitch (for $h \ll 1$)

$$\gamma_z/j \approx -4(h^2/j^2 - \tfrac{1}{8}) \cos\beta_0 , \qquad (23)$$

lying underneath the boundary curve of stability with $h^2/j^2 > \frac{7}{40} \approx 0.175$.

For an extensive study of stability questions, the reader is referred to ref. [10].

The author is grateful to Profs. Maki, Mermin, Hall, and Fetter for many useful discussions.

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