BEYOND LANDAU'S THEORY OF FERMI LIQUIDS

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Landau's theory of density and pair fluctuations in a Fermi liquid is extended to include also fluctuations in the vertex functions, i.e. in the Landau parameters themselves. This is a natural consequence of the higher effective action $\Gamma[G, \alpha]$ which depends explicitly on the full propagator G and the vertex α and whose extrema in G and α determine physical configurations of these quantities.

Effective actions are the ideal tool for the discussion of quantum phenomena in terms of classical variables. Recently [1], we stressed the relevance of the higher effective action $\Gamma[G, \alpha]$ with respect to Fermi systems, which depends explicitly on the Green's function G and the vertex function α, both fully interacting. Physical G, α configurations are obtained by extremizing $\Gamma[G, \alpha]$ in both variables. In the case of an instantaneous potential, only the equal-time part of G is involved such that G collects the density matrix $\rho(t) = \langle \psi^+(t)\psi(t)\rangle$ and the pair correlation $\Delta = \langle \psi(t)\psi(t)\rangle$ of the system as

$$G(t, t) \equiv \begin{pmatrix} \Delta & 1-\rho^{\mathrm{T}} \\ \rho & \Delta^+ \end{pmatrix} = \langle \varphi(t)\varphi(t')\rangle|_{t'=t-\epsilon} . \tag{1}$$

Notice that we are using a doubled field notation

$$\varphi \equiv \begin{pmatrix} \psi \\ \psi^+ \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi^+ , \tag{2}$$

which combines the standard creation and annihilation operators in one quasi-real symbol, such that the fundamental action has the form

$$\mathcal{A}[\varphi] = \tfrac{1}{2}\varphi(\mathrm{i}G_0^{-1} + K^{\mathrm{cp}})\varphi - \tfrac{1}{8}V\varphi\varphi\varphi\varphi , \tag{3}$$

where all orbital and time indices have been omitted, the matrix

$$\mathrm{i}G_0^{-1} = \begin{pmatrix} 0 & \mathrm{i}\partial_t + \epsilon \\ \mathrm{i}\partial_t - \epsilon & 0 \end{pmatrix} \tag{4}$$

is the inverse of the free propagator, $K^{\mathrm{cp}} = \left(\begin{smallmatrix} 0 & -\mu \\ \mu & 0 \end{smallmatrix}\right)$ is the chemical potential in 2×2 form, and V is the interaction potential symmetric in the four doubled indices contracted with φ.

In ref. [1] we have used the generating functional of the theory with bi- and quadrulocal sources K and L:

$$W[K, L] \equiv -\mathrm{i}\log\langle T\exp[\tfrac{1}{2}\mathrm{i}\varphi K\varphi - (\mathrm{i}/4!)L\varphi\varphi\varphi\varphi]\rangle , \tag{5}$$

and introduced $\Gamma[G, \alpha]$ as the Legendre transformed functional

$$\Gamma[G, \alpha] \equiv W[K, L] - W_K K - W_L L , \tag{6}$$

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where G and α are defined as the connected Green's function G and the four-particle vertex function, respectively, which are related to the derivatives of $W[K,L]$ as

$$W_K = \tfrac{1}{2}G\,, \quad W_L = -(1/4!)(-\mathrm{i}\alpha G^4 + 3G^2)\,. \tag{7,8}$$

By construction, $\Gamma[G,\alpha]$ is extremal on physical G, α configurations and is therefore the appropriate effective action of the system.

There is a simple expansion of $\Gamma[G,\alpha]$ in powers of the fully interacting vertex or, equivalently, in the number of exact fermion loops (quasi-classical expansion). This reads (see ref. [1] for details)

$$\Gamma[G,\alpha] = -\tfrac{1}{2}\mathrm{i}\ \mathrm{tr}\log G^{-1} + \tfrac{1}{2}\ \mathrm{tr}[(\mathrm{i}G_0 + K^{\mathrm{cp}})G] - \tfrac{1}{8}VG^2 + \tfrac{1}{48}\mathrm{i}(2\alpha V - \alpha^2)G^4 + \tfrac{1}{48}\alpha^3 G^6 + \frac{1}{8\cdot 16}\mathrm{i}\alpha^4 G^8 - \dots\,. \tag{9}$$

The equations of motion following from extremizing Γ are

$$\mathrm{i}G^{-1} = \mathrm{i}G_0^{-1} + K^{\mathrm{cp}} - \tfrac{1}{2}VG + \tfrac{1}{6}\mathrm{i}\alpha G^3 V\,, \tag{10}$$

$$V = V[G,\alpha] = \alpha + \tfrac{3}{2}\mathrm{i}\alpha^2 G^2 - \tfrac{3}{4}\alpha^3 G^3 + \dots\,. \tag{11}$$

To lowest (two-loop) approximation, $V = \alpha$ from (11), and eq. (10) reduces to the Hartree–Fock–Bogoliubov equation

$$G = \mathrm{i}[\mathrm{i}G_0^{-1} + K^{\mathrm{cp}} - \tfrac{1}{2}VG]^{-1}\,. \tag{12}$$

The new and distinctive feature of (11) lies in the possibility of a spontaneous generation of vertices which are absent in the original potential [1]. This is of particular importance for nuclei where V has only a particle-conserving $\psi^+\psi\psi^+\psi$ part while α carries strong four-particle correlations of the type $\psi\psi\psi\psi$.

Suppose now that we have found a static solution of eqs. (10), (11). We may then study small oscillations around this and substitute

$$G \to G + \delta G\,, \quad \alpha \to \alpha + \delta\alpha\,. \tag{13}$$

The action changes quadratically by an amount

$$\delta^2\Gamma[G,\alpha] = \tfrac{1}{4}\mathrm{i}\ \delta G\, G^{-1} \times G^{-1}\,\delta G - \tfrac{1}{8}V\,\delta G\,\delta G + \tfrac{1}{8}\mathrm{i}(\alpha G^2\,\delta G\,\delta G\,V - \tfrac{1}{3}\alpha G^3\,\delta G\,V_G\,\delta G)$$
$$- \tfrac{1}{48}\mathrm{i}\ \delta\alpha\, G^4\, V_\alpha\,\delta\alpha + \tfrac{1}{12}\mathrm{i}(\delta\alpha\, G^3\,\delta G\,V - \alpha G^3\,\delta G\,V_\alpha\,\delta\alpha)\,. \tag{14}$$

The index contractions are shown graphically in fig. 1.

It is obvious that this represents the field-theoretic derivation and natural generalization of Landau's free-energy expansion for Fermi liquids which was obtained by him on phenomenological grounds. In order to see this, consider a translationally invariant solution G, α. Then the matrix on the right-hand side of eq. (10) has a complete set of eigenstates $\chi^{\boldsymbol{p}}(t)$ with eigenvalues $\kappa(\boldsymbol{p})$ which are labelled by the spatial momentum $\boldsymbol{p}$. Of course, these are just the single-particle orbits in the self-consistent background of all the other particles. They have to be filled up to some Fermi level of momentum p_{F} and we may define Fermi velocity $v_{\mathrm{F}} = |\boldsymbol{v}|_{p_{\mathrm{F}}}$ $[\boldsymbol{v} \equiv \partial\kappa(p)/\partial\boldsymbol{p}]$ and the effective mass as $m^* = p_{\mathrm{F}}/v_{\mathrm{F}}$. Further we take V to be time independent and instantaneous such that we may deal with equations for the equal-time Green's function (1) only. Then the first term in our expansion (14) of the action can be evaluated in the basis $\chi^{\boldsymbol{p}}(t)$ as follows:

$$-\mathrm{i}G(p+q/2)G(p-q/2) = \mathrm{i}\int \frac{\mathrm{d}\epsilon}{2\pi}\,\frac{1}{\epsilon + q_0/2 - \kappa(\boldsymbol{p}+\boldsymbol{q}/2)}\,\frac{1}{\epsilon - q_0/2 - \kappa(\boldsymbol{p}-\boldsymbol{q}/2)}$$
$$= [q_0 - \kappa(\boldsymbol{p}+\boldsymbol{q}/2) + \kappa(\boldsymbol{p}-\boldsymbol{q}/2)]^{-1}\int\frac{\mathrm{d}\epsilon}{2\pi}\left(-\frac{\mathrm{i}}{\epsilon + q_0/2 - \kappa(\boldsymbol{p}+\boldsymbol{q}/2)} + \frac{\mathrm{i}}{\epsilon - q_0/2 - \kappa(\boldsymbol{p}-\boldsymbol{q}/2)}\right)$$
$$= [q_0 - \kappa(\boldsymbol{p}+\boldsymbol{q}/2) + \kappa(\boldsymbol{p}-\boldsymbol{q}/2)]^{-1}\,[n(\boldsymbol{p}+\boldsymbol{q}/2) - n(\boldsymbol{p}-\boldsymbol{q}/2)]\,, \tag{15}$$

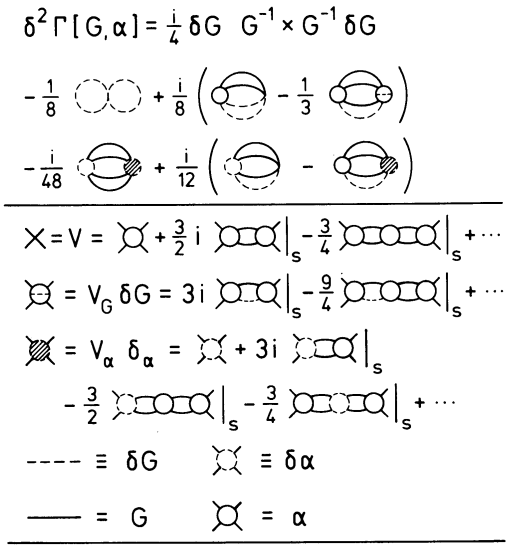

Fig. 1. The quadratic variation $\delta^2\Gamma[G, \alpha]$ of the effective action around stationary solutions of the Green's function G and the vertex function α. The deviations are denoted by δG and $\delta\alpha$. The first two pieces correspond to Landau's expansion of the free energy in terms of density oscillations. The others contribute higher collision integrals and introduce vertex oscillations. The new coupled eigenmodes follow from extremizing $\delta^2\Gamma[G, \alpha]$. The symbol $|_s$ denotes antisymmetrization in the external legs.

where q is the total and p the relative momentum of the bilocal quantity, q_0 is the total energy, and $n(\boldsymbol{p})$ is the Fermi distribution function. In the long-wavelength limit (15) reduces to

$$-\mathrm{i}G \times G = (\partial n/\partial\kappa)\boldsymbol{v}\cdot\boldsymbol{q}/(q_0 - \boldsymbol{v}\cdot\boldsymbol{q})\,, \tag{16}$$

such that the first term in (14) reads

$$\tfrac{1}{4}\delta G\,(q_0/\boldsymbol{v}\cdot\boldsymbol{q} - 1)(\partial n/\partial\kappa)^{-1}\delta G\,, \tag{17}$$

with the lowest-order equation of motion becoming from (13)

$$(q_0 - \boldsymbol{v}\cdot\boldsymbol{q})\delta G = \boldsymbol{v}\cdot\boldsymbol{q}(\partial n/\partial\kappa)(V/2)\delta G\,, \tag{18}$$

which is just Landau's transport equation [2].

The higher terms in our expansion determine, one the one hand, corrections to (18) due to collision integrals. On the other hand, and this is the novel feature of this approach, they introduce oscillations in the vertex functions, i.e. in the Landau interaction parameters themselves. The detailed study of the coupled equations of motion which extremize (14) promises to yield new insights into the dynamic properties of higher correlation functions as well as into more involved fluctuation phenomena.

References

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[2] D. Pines and P. Nozières, The theory of Fermi liquids I (Benjamin, New York, 1966) eq. (1.79).